Doctoral Qualifying Examination

in

Applied Mathematics

Part A: Analysis

February 3, 1997

Problem 1. Give an example of each of the following:

(A) A function continuous at the origin but not differentiable there.

(B) A sequence of continuous functions converging pointwise to a discontinuous function.

(C) A sequence of nonnegative, continuous functions $\{f_n\}$ converging pointwise to zero such that

$$\int_0^1 f_n(x)\,dx \; = \; 1 \qquad \text{for all } n.$$

Problem 2. Let $\mathbf{Q}$ be the set of rational numbers. Suppose $f : [2,3] \to (-\infty, +\infty)$ is given by

$$f(x) \; = \; \begin{cases} 0 & \text{for } x \in \mathbf{Q} \cap [2,3] \\ (x-2)^{-1/4} & \text{for } x \in [2,3] - \mathbf{Q} \end{cases}$$

(a) Is f Riemann integrable on $[2,3]$? Explain.

(b) Evaluate the Lebesgue integral $\int_2^3 f(x)dx$.

(c) Find $\lim_{n\to\infty} \int_2^3 (1 + (x-2)^{1/4})^{-n}\,dx$. Justify your calculation.

Problem 3.

(A) Let $f(x) = \pi - |x|$ on $[-\pi, \pi]$ and $f(x + 2\pi) = f(x)$. Find the Fourier series of f and its sum. Show that

$$\sum_{n=0}^{\infty} (2n+1)^{-4} = \pi^4/96.$$

(B) Let $f(x) = |x|^{1/4}$ on $[-1, 1]$ and be periodic of period 2 on $(-\infty, \infty)$. Evaluate the sum of its Fourier series (do not look for the Fourier coefficients).

Problem 4. Consider the complex-valued function f of the complex variable z defined by

$$f(z) = \begin{cases} \exp(-1/z^4) & z \neq 0 \\ 0 & z = 0 \end{cases}$$

(A) Show that f satisfies the Cauchy-Riemann equations at every point in the complex plane including $z = 0$.
(B) What is the largest subset of the complex plane where f is analytic? Continuous?
(C) Is $|f(z)|$ bounded on the unit disk $|z| \leq 1$?

Problem 5. Evaluate

$$\int \frac{1}{(z^2+4)^2}\, dz \qquad \text{and} \qquad \int \frac{\sin z}{z}\, dz$$

on the contour $|z - i| = 2$ traced in the positive sense.

Problem 6.

(A) Derive the Taylor series for $\frac{1}{1-z}$ about $z = i$. Sketch the region where the series converges.
(B) Discuss the convergence of

$$\sum_{n=0}^{\infty} (1 + z^{1/2})^n.$$