DOCTORAL QUALIFYING EXAM IN ANALYSIS

February 15, 1996

(3 hours)

1. Let $f_n : \mathcal{R} \to \mathcal{R}$ be continuous, $f_n \in L(\mathcal{R})$ and $f_n \overset{a.e}{\to} f$ on $\mathcal{R}$.. Find $\lim_{n\to\infty} \int_{\mathcal{R}} e^{-|x|} \cos(f_n(x))dx$.

2. Let $r > 0$ and $g \in C[0,r]$. Show that the following integral equation has a unique solution in $C[0,r]$:

$$f(x) = g(x) + \int_0^x xye^{-xy^2} f(y)dy.$$

3. (a) Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \sin(1+\frac{x}{n})$ converges uniformly for $|x| \le 1$. Prove that it converges pointwise
on $\mathcal{R}$. (Hint: use $1 + \frac{x}{n} \in [0, \pi/2)$ for $n > 1$).

(b) If $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \sin(1+\frac{x}{n})$, show that f is differentiable on $\mathcal{R}$ and

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}} \cos(1+\frac{x}{n}).$$

4. Let $f(x) = |x|^{1/2}$ on $[-\pi, \pi]$ and be 2π-periodic. Find the sum of the Fourier series of f for each
$x \in [0, 2\pi]$.

5. Describe the branches and the Riemann surface of the multivalued function $\log\sqrt{z-1}$.

6. Solve the Dirichlet problem

$$\begin{aligned} \triangle\varphi &= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} \quad (0 < \arg z < \frac{\pi}{3}) \\ \varphi &= 0 \; for \arg z = 0, \; \varphi = 1 \; for \arg z = \frac{\pi}{3}, \end{aligned}$$

where $z = x + iy$, by conformally transforming it to a problem on the half-plane $y > 0$.

7. For the function $f(z) = \frac{1}{z^2(z^2+i)}$

(a) Find and classify all singularities and compute the residues.
(b) Find the Laurent expansion for $0 < |z| < 1$.
(c) Find the Laurent expansion for $1 < |z|$.

8. Compute the following integrals:

$$(a) \quad \int_0^\infty \frac{\cos x}{x^2+1} dx \qquad\qquad (b) \quad \int_1^\infty \frac{dx}{x\sqrt{x^2-1}} \qquad\qquad (c) \quad \int_0^\infty \frac{\log x}{x^2+1} dx.$$