Doctoral Qualifying Examination

in

Applied Mathematics

Part A: Analysis

January 23, 1995

Problem 1. For each series below determine the complex values of z for which it converges.

$$\text{(A)} \sum_{n=1}^{\infty} \frac{z^n}{n!} \qquad \text{(B)} \sum_{n=1}^{\infty} z^n \qquad \text{(C)} \sum_{n=1}^{\infty} n! z^n \qquad \text{(D)} \sum_{n=1}^{\infty} \left(\frac{1+z}{1-z}\right)^n$$

Problem 2. Suppose $f_n : (a, b) \to \mathbf{R}$ is differentiable $\forall n \in \mathbf{N}$ and that f_n converges uniformly to a limit function f.

(A) Show (by example) that f may not be differentiable on (a, b).

(B) Show (by example) that even if f is differentiable f_n' may fail to converge pointwise to f'.

(C) Show that the function F defined by

$$F(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n \log(n+1)}$$

exists and is differentiable for $x \in (0, 2\pi)$.

(D) Suppose that $h : \mathbf{R}^2 \to \mathbf{R}$ has continuous partial derivatives of order one. Suppose that $g : [0, 1] \to \mathbf{R}$ is Lebesgue integrable. Define $F : \mathbf{R} \to \mathbf{R}$ by

$$F(x) = \int_0^1 h(x, y) g(y)\, dy.$$

Show that F is differentiable on $\mathbf{R}$ and that

$$F'(x) = \int_0^1 \frac{\partial h}{\partial x}(x, y) g(y)\, dy.$$

Problem 3. Let f_1 and g_1 be continuous complex-valued functions defined on $[0, 1]$. Consider the sequences $\{f_n\}$ and $\{g_n\}$ given by

$$f_{n+1}(x) = \sin x + \frac{1}{2} \int_0^x g_n(s)\, ds,$$

$$g_{n+1}(x) = \cos x + \int_0^1 f_n(s) e^{ixs}\, ds.$$

Show that there exist continuous functions f and g defined on $[0, 1]$ such that $f_n \to f$ and $g_n \to g$ pointwise.

Problem 4. Prove the Riemann-Lebesgue lemma which states that if f is Lebesgue integrable on the interval I then

$$\lim_{y\to\infty}\int_I f(x)e^{iyx}\,dx \;=\; 0.$$

Problem 5. Discuss the singularities of

$$w \;=\; \frac{1}{\sqrt{1+\sqrt{z}}}.$$

Problem 6. Discuss $w=\cosh^{-1}(z)$ and calculate dw/dz.

Problem 7. Evaluate the following integrals using contour integration:

$$\text{(A)}\ \int_0^\infty \frac{dx}{x^2+3x+2}, \qquad \text{(B)}\ \int_0^\infty \frac{\cos ax}{1+x^2}\,dx \quad a\ge 0, \qquad \text{(C)}\ \int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} \quad a>b\ge 0.$$

Problem 8. Consider an nth degree polynomial

$$P(z) \;=\; z^n + \; a_{n-1}z^{n-1} \;+\; a_{n-2}z^{n-2} \;+\; \ldots \;+\; a_1 z \;+\; a_0$$

with n distinct roots $\{z_1, z_2, \ldots, z_n\}$.

(A) Suppose $R>\max\{|z_1|,|z_2|,\ldots,|z_n|\}$. Use residues to compute

$$\frac{1}{2\pi i}\oint_{|z|=R}\frac{zP'(z)}{P(z)}\,dz$$

in terms of the roots of P.

(B) Use the parameterization $z=Re^{i\theta}$ to convert the contour integral in (A) to a definite integral over the interval $[0,2\pi]$. For large R evaluate this definite integral in terms of the coefficients of the polynomial.

(C) What well-known relationship between the coefficients and the roots of a polynomial follows from the results of (A) and (B)?