Ph.D Qualifying Exam in Analysis

January 18, 2002

Problem 1.

Let $\{x\}$ denote the distance from x to the nearest integer.

(a) Graph $\{x\}$.

(b) Let $f_n(x) = \{10^n x\}/10^n$. Show that $\Sigma_{n=1}^{\infty} f_n(x)$ converges uniformly to a continuous function.

Problem 2.

Let a_{mn} be positive numbers. Prove $\Sigma_{n=1}^{\infty}\Sigma_{m=1}^{\infty} a_{mn} = \Sigma_{m=1}^{\infty}\Sigma_{n=1}^{\infty} a_{mn}$.

Problem 3.

Let $\{\phi_n\}$ be orthonormal on I. Suppose $\Sigma|c_n|^2$ converges for any sequence of complex numbers $\{c_n\}$. Let $s_n = c_1\phi_1 + \ldots + c_n\phi_n$. Prove there exists a function $f \in L^2(I)$ such that $\{s_n\}$ converges to f in $L^2(I)$, and such that $f \sim \Sigma_{n=1}^{\infty} c_n\phi_n$.

Problem 4.

Define a branch of $\sqrt{z^2-1}$ that is analytic except for the interval $[-1,1]$ on the real axis in the complex plane and determine several terms in the Laurent series expansion of this branch valid for $|z| > 1$.

Problem 5.

(a) Let f be analytic on an open set U in the complex plane. Suppose $z_0 \in U$ and $f'(z_0) \neq 0$. Use the principle of the argument and Rouché's theorem to prove that

$$\hat{f}(w) = \frac{1}{2\pi i}\int_{C_\rho} \frac{\zeta f'(\zeta)d\zeta}{f(\zeta) - w},$$

where $C_\rho = \{z \in \mathbf{C} : |z - z_0| = \rho > 0\}$ and ρ is sufficiently small, represents the inverse of f in a sufficiently small neighborhood of z_0, which is defined and analytic in some neighborhood of $w_0 = f(z_0)$.

(b) Use the result in (a) to find a power series representation of the inverse of $f(z) = 2z + z^4$ on a neighborhood of $z = 0$.

Problem 6.

Compute each of the following integrals:

$$(i)\ \int_0^{\pi} \frac{d\theta}{1+\sin^2\theta} \qquad (ii)\ \int_0^{\infty} \frac{\cos x}{1+x^4}dx \qquad (iii)\ \int_0^{\infty} \frac{\log(ax)}{x^2+1}dx \quad (0 < a).$$