Analysis Qualifying Exam. June 15, 2004
Answer ALL questions and show work clearly

1. Let X be a finite closed interval $[a, b]$ in $\mathbf{R}$, let $\mathbf{X} = \mathbf{B}$, the collection of Borel sets, and let λ be Lebesgue Measure. If f is a continuous function on X, show that

$$\int f \, d\lambda = \int_a^b f \, dx$$

2. (a) Give an example of a function which converges almost everywhere that fails to converge in measure.

   (b) Let f_n and f be real-valued and $(X, \mathbf{X}, \mu)$ be a measure space. Let $f_n \to f$ almost everywhere. Prove that $f_n \to f$ in measure if f_n is dominated by an integrable function.

3. Suppose that the series $\frac{1}{2}a_0 + \Sigma_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ converges uniformly on $[-\pi, \pi]$. Prove that there exists a continuous function f on $[-\pi, \pi]$ such that the fourier series of $f \sim \frac{1}{2}a_0 + \Sigma_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$.

4. (a) Expand each of the following functions $f(z)$ in a Laurent series on the indicated domain:

   (i)

$$f(z) = \frac{1}{(z-1)(z-2)},$$

   $1 < |z| < 2$.

   (ii)

$$f(z) = \ln \frac{z-1}{z-2},$$

   $|z| > 2$.

   (b) Calculate $Res_{z=a} f(z)g(z)$ (the residue at $z = a$) given that $f(z)$ is analytic at $z = a$ if

   (i) $g(z)$ has a simple pole with residue A at $z = a$

   (ii) $g(z)$ has a pole of order k with principal part

$$\frac{a_{-1}}{z-a} + \cdots + \frac{a_{-k}}{(z-a)^k}$$

   at $z = a$.

5. (a) Assume $f(z)$ is analytic except at a finite number of isolated singular points. Also assume the residue of $f(z)$ at infinity is zero. Show the sum of all the residues of $f(z)$ equals zero.

(b) Use part (a) to compute (avoiding excessive calculation)

$$\int_C \frac{dz}{(z-3)(z^5-1)}$$

where C is the circle $|z| = 2$.

6. (a) Evaluate the integral

$$I = \int_0^\infty \frac{\cos(\lambda x)}{x^2 + a^2} dx,$$

where $a > 0,\ \lambda > 0$.

(b) Let $f(z) \neq \text{const}$ be analytic on $|z| < 1$ and continuous on $|z| \leq 1$, and suppose $|f(z)|$ has the same value at all points on the boundary $|z| = 1$. Prove that $f(z)$ has at least one zero at a point of $|z| < 1$. (You may use the fact that if a differentiable function has a constant modulus on an open connected set, then f itself is constant.)