PhD Qualifying Examination
Part A: Analysis
January 15-16, 2004

1. A sequence of functions $\{f_n\}$ defined on a set E in a metric space X with metric d, is said to be equicontinuous on E if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ whenever $d(x,y) < \delta$ for $x, y \in E$, for each $n \in \mathbf{Z}^+$. Let $C(K)$ denote the set of continuous functions on a compact metric space K. Prove that if $f_n \in C(K)$ for each $n \in \mathbf{Z}^+$ and if f_n converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.

2. Prove the Riemann-Lebesgue Lemma: If I is any interval of $\mathbf{R}$ and $f \in L(I)$, then for each $\beta \in \mathbf{R}$,

$$\lim_{\alpha \to \infty} \int_I f(t) \sin(\alpha t + \beta)\, dt = 0.$$

3. (a) Let $f_n = n\chi_{[1/n, 2/n]}$. Show that $f_n \to 0$ for all x, but does not converge to 0 in $L_p(\mathbf{R}, \mathbf{B}, \lambda)$.
   (b) Let $(X, \mathbf{X}, \mu)$ be a measure space. Suppose $f_n \to f$ almost everywhere where $f_n \in L_p$ and f is measurable. Prove that if there exist $g \in L_p$ such that $|f_n| \leq g$ for all n and x, then $f \in L_p$ and $f_n \to f$ in L_p.
   (c) Does the result in part (a) contradict the result in part (b)?

4. Consider the real-valued function $u(x,y) = x^3 - 3xy^2 + 2y$.
   (a) Show that $u(x,y)$ can be the real part of an analytic function, $f(z)$.
   (b) Determine the imaginary part of this analytic function, $v(x,y)$, where $v(-1,1) = -2$.
   (c) Determine $f(z)$.
   (d) Determine the integral

   $$\oint_C \frac{f(z)}{z+1} dz$$

   where C is the curve described in polar coordinates by $r = 1 + \cos\theta$, $\quad 0 \leq \theta < 2\pi$ and is traversed in the direction of increasing θ.

5. (a) Determine if the following statement is true or false: If f is analytic at each point of a closed contour Γ, then

   $$\oint_\Gamma f(z) dz = 0.$$

   If this is true, state why, and if it is false, give a counter-example.
   (b) Let the nonconstant function f be analytic in the bounded domain D and continuous up to and including the boundary B. Prove that if $|f(z)|$ is constant on B, then f must have at least one zero in D.

6. Use complex variables techniques to evaluate the following integrals:
   (a) $\oint_\Gamma \frac{1}{z^2+1} dz$, where Γ is:
      i. the positively oriented circle of radius $\sqrt{2}$ centered at (1,1) in the complex plane,
      ii. the negatively oriented circle of radius 1 centered at (0,-1), and
      iii. the positively oriented circle of radius 2 centered at the origin.
   (b) $\oint_\Gamma \frac{\cos\, 2z}{(z-\pi)^3}\, dz$, where Γ is a positively oriented square centered at the origin with sides of length 7.
   (c) $\int_{-\infty}^{\infty} \frac{e^{-i2x}}{x-3} dx$.