Doctoral Qualifying Exam: Analysis

Tuesday, June 12, 2007

1. Suppose that $f_n \in L^1([0,1])$ for $n \in \mathbf{N}$ and that f_n converges to f uniformly on $[0,1]$. Prove that $f \in L^1([0,1])$ and that
$$\lim_{n\to\infty} \int_0^1 f_n\,dx \;=\; \int_0^1 f\,dx.$$

2. (a) State and prove the Riemann-Lebesgue Lemma.

   (b) Prove
$$\lim_{n\to\infty} n^2 \int_0^{2\pi} e^{inx} x^2 (2\pi - x)^2\,dx \;=\; 0.$$

3. Let $y_0(x) = \sin x$. For $n \in \mathbf{N}$ let $y_n : [0,2\pi] \to \mathbf{R}$ be defined by
$$y_n(x) \;=\; \frac{1}{\pi^2} \int_0^{2\pi} \sin(x-\xi) y'_{n-1}(\xi)\,d\xi.$$

   (a) Prove that y_n converges uniformly to continuous function y.

   (b) Find y.

4. Consider the integral
$$\int_0^\infty \frac{x^\alpha}{1+x^2}\,dx.$$
For what values of α does the integral converge? Evaluate the integral using a circular contour with an appropriate cut. Give details, especially all necessary estimates.

5. (a) Consider the conformal map
$$B_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}.$$
What is the image of the unit disc under this mapping?
(b) Suppose that f is analytic and bounded by 1 in the unit disc and that $f(1/2) = 0$. Derive the bound $|f(\frac{3}{4})| \le \frac{2}{5}$. (Hint: consider
$$g(z) = \begin{cases} f(z)/B_{1/2}(\mathrm{z}) & z \neq 1/2 \\ \frac{3}{4} f'(1/2) & z = 1/2 \end{cases}$$
and show that $g(z)$ is analytic in $|z| < 1$. Next find a bound on $|g|$ by considering the limit $|z| \to 1$. Finally, use the bound on g to obtain a bound on f.)
(c) Find a function f that satisfies the assumptions in (b) and for which the bound in (b) is sharp, i.e., $|f(\frac{3}{4})| = \frac{2}{5}$.

6. Suppose that f is bounded and analytic in $Im\ z \ge 0$ and real on the real axis. Prove that f is constant.