Doctoral Qualifying Exam:
Real and Complex Analysis
Thursday, January 13, 2000

1. Consider the function f defined on R^1 by

$$f(x) = \begin{cases} 0 & \text{(x irrational)}, \\ \frac{1}{n^2} & (x = m/n \text{ in lowest terms}) \end{cases}$$

with $f(0) \equiv 0$.

(a) Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point except 0.

(b) Is f Riemann integrable on [0,1]? (Justify your answer.)

(c) Is f Lebesgue integrable on [0,1]? (Justify your answer.)

(d) If your answer in (b) or (c) is yes, then evaluate the integral.

(e) Find all x for which $f(x)$ is differentiable.

2. (a) Show that

$$\cos\ x + \ldots + \cos\ nx = \frac{\sin\ (n+\frac{1}{2})x - \sin\ \frac{x}{2}}{2\ \sin\ \frac{x}{2}}.$$

(b) Prove that if a_n is decreasing with $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{n=\infty} a_n \cos\ nx$ converges provided $x \neq 2k\pi$, where k is an integer.

3. (a) Let $\{f_n\}$ and $\{g_n\}$ be sequences of bounded functions which converge uniformly on a set E. Show that the sequence $\{f_n g_n\}$ converges uniformly on E.

(b) After removing the boundedness assumption in part (a), provide a counterexample for which the sequence $\{f_n g_n\}$ does not converge uniformly.

4. Evaluate the following quantities using contour integration in the complex plane:

(a)

$$\int_0^\infty \frac{(\ln x)^2}{x^2+4}\, dx$$

(b)

$$\int_0^{2\pi} e^{\cos\theta} \cos\,(n\theta - \sin\theta)\, d\theta$$

where n is a positive integer

5. Given $f(z) = \sqrt{z}/(1+\sin z)$, find the Laurent expansion in some annular region about $z = \pi/2$. Classify each singularity of this function, including the point at infinity.

6. (a) Let $f(z)$ be analytic in a domain D. Show that if the conjugate of f is analytic in D, then $f(z)$ is constant in D.

   (b) State the maximum modulus theorem. Find the maximum modulus of $f(z) = z^2 + 2z + 3\,i$ for $|z| \leq 1$.

   (c) In which quadrants do the roots of $f(z) = z^4 + z^3 + 4z^2 - 2z + 3$ lie?