Doctoral Qualifying Exam: Real Analysis and Probability

Tuesday, June 12, 2007

1. Suppose that $f_n \in L^1([0,1])$ for $n \in \mathbf{N}$ and that f_n converges to f uniformly on $[0,1]$. Prove that $f \in L^1([0,1])$ and that

$$\lim_{n\to\infty} \int_0^1 f_n \, dx \;=\; \int_0^1 f \, dx.$$

2. (a) State and prove the Riemann-Lebesgue Lemma.

   (b) Prove

$$\lim_{n\to\infty} n^2 \int_0^{2\pi} e^{inx} x^2 (2\pi - x)^2 \, dx \;=\; 0.$$

3. Let $y_0(x) = \sin x$. For $n \in \mathbf{N}$ let $y_n : [0, 2\pi] \to \mathbf{R}$ be defined by

$$y_n(x) \;=\; \frac{1}{\pi^2} \int_0^{2\pi} \sin(x - \xi) y'_{n-1}(\xi) \, d\xi.$$

   (a) Prove that y_n converges uniformly to continuous function y.

   (b) Find y.

4. A random variable (r.v.) X and a sequence of r.v.s $\{X_n;\ n = 1, 2, \cdots\}$ are defined on a common probability space $(\Omega, \mathcal{F}, P)$.

   (a) Suppose we are given that

$$P\{\omega :\ (X_n, X) = (0,1)\} = P\{\omega :\ (X_n, X) = (1,0)\} = \frac{1}{2}.$$

   Examine if X_n converges to X in any of the convergence modes $\xrightarrow{P}$, $\xrightarrow{a.s.}$, $\xrightarrow{d}$, and $\xrightarrow{L^r}, r > 0$.

   (b) Prove that $X_n \xrightarrow{P} X \Longrightarrow X_n \xrightarrow{d} X$. Show by a counterexample that the converse is, in general false.

   (c) Prove that the converse is true, if X is degenerate ($X_n \xrightarrow{d} c \Longrightarrow X_n \xrightarrow{P} c$, where c is a real constant.)

5. (a) X is a r.v. with $EX = 0$ and $\mathrm{var}(X) = \sigma^2 < \infty$. Use the Markov inequality for the r.v. $h(X) := (X + c)^2$, $c > 0$ for suitable choices of c, to prove that,

$$\begin{aligned} P(X > x) &\le \tfrac{\sigma^2}{\sigma^2 + x^2}, \quad \text{if } x > 0 \\ P(X > x) &\ge \tfrac{x^2}{\sigma^2 + x^2}, \quad \text{if } x < 0 \end{aligned}$$

(b) If $M(t) := E(e^{tX})$ exists for all $t > 0$, then show that for all real s, we have

$$\sup_{t>0} P(tX > s^2 + \ln M(t)) \leq e^{-s^2}.$$

6. Suppose $X_1, X_2, \cdots$ are i.i.d. L^2.

(a) Prove that

$$Y_n := \frac{2}{n(n+1)} \sum_{j=1}^{n} jX_j \xrightarrow{P} EX_1.$$

(b) Using Kolmogoroff's convergence criterion, and Kronecker's lemma, show that the result in (a) can be stregthened to $Y_n \xrightarrow{a.s.} EX_1$.

(c) Which sequence of r.v.s obeys WLLN and SLLN, in view of the results (a) and (b) above ?