Doctoral Qualifying Exam: Applied Mathematics

Friday, March 23, 2007

1. A spherical, thermally conducting particle with a constant radius R is released in a Stokesian fluid. The liquid has a constant kinematic viscosity ν_0, constant density ρ_0 and a constant temperature T_0 (the liquid is highly thermal-conducting, and a constant heat bath keeps the temperature at T_0 for all time.) Upon release in the liquid, the spherical particle is initially at temperature $T_i < T_0$. The density of the spherical particle is observed to be dependent only on the average temperature, $\rho_s = \beta\rho_0 \left(\frac{\langle T \rangle}{T_0}\right)^\alpha$, where $\langle T \rangle \equiv \int_0^R T dr/R$ is the average temperature with constant coefficients $1 \geq \beta > 0$ and $\alpha > 1$. Gravity acts downwards, with acceleration g.

   (a) The density contrast between the sphere and the background fluid gives rise to a buoyancy force on the particle. In addition, if we assume that the motion of the particle obeys Stokes' law, the fluid also exerts a drag force on the particle which is $6\pi\rho_0\nu_0 R$ multiplied by the velocity of the particle relative to the fluid. Set up the balance of forces on the particle and apply Newton's second law to find the second-order ODE for the particle displacement $x(t)$ from the initial release location.

   (b) The buoyancy force in (a) is a function of the average temperature, which can be computed from particle temperature $T(r,t)$. Assuming that the particle thermal conductivity κ is large and the convective flux is negligible, write down the heat equation in spherical coordinate for the particle temperature $T(r,t)$ ($\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$.) What are the boundary conditions for T? Describe the evolution of particle temperature T. As the particle thermalizes with the fluid, argue that the particle will be moving at a constant terminal velocity U. What is this terminal velocity U?

2. Consider the eigenvalue problem

$$\frac{d}{dx}\left(x^2 \frac{du}{dx}\right) + \lambda u = 0, \quad 1 < x < e,$$

   subject to the homogeneous boundary conditions

$$u(1) = 0 = u(e)$$

   where λ is the eigenvalue.

   (a) Show that the eigenvalues must be positive.

   (b) Obtain the exact eigenvalues, λ_n, and corresponding eigenfunctions $u_n(x; \lambda_n)$.

3. Consider the periodic boundary-value problem given by

$$\frac{d^2u}{dx^2} + u = f(x), \quad 0 < x < 2\pi,\ u(0) = u(2\pi),\ \frac{du}{dx}(0) = \frac{du}{dx}(2\pi)$$

where $f(x)$ is a given function. Determine the appropriate Green's function for this problem.

4. Let D be the exterior of the surface S given by $x^2 + y^2 + \frac{z^2}{9}$ and consider the boundary value problem

$$\nabla^2 u = 0, \qquad \mathbf{x} \in \mathbf{D}$$
$$\frac{\partial u}{\partial n} = f(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{S}$$
$$u \to 0, \qquad r = |\mathbf{x}| \to \infty.$$

(a) Show that $u \sim A/r$ as $r \to \infty$.

(b) Express the constant A in terms of the boundary data f.

5. Solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1$$
$$\frac{\partial u}{\partial x} - hu = 0, \quad x = 0, \qquad \frac{\partial u}{\partial x} = 0, \quad x = 1$$
$$u(x, 0) = f(x), \qquad 0 < x < 1.$$

where h is a constant. Find the behavior of u as $t \to \infty$ when $h > 0$ and when $h < 0$.

6. Find the Green's function satisfying

$$\nabla^2 G = \delta(x - x')\,\delta(y - y'), \quad y > 0, \quad |x| < \infty$$
$$\frac{\partial G}{\partial y} + \alpha G = 0, \quad y = 0, \quad |x| < \infty.$$

Investigate the behavior of your solution in the two limits: $\alpha \to 0$, and $\alpha \to \infty$.