Doctoral Qualifying Exam: Applied Mathematics

Wednesday, January 10, 2007

1. Consider a room of volume V, containing at time t an amount $Q(t)$ of carbon monoxide, well mixed throughout the room with a concentration $c(t)$, where $c(t) = Q(t)/V$. Assume that at time $t = 0$ cigarette smoke, containing a concentration k of carbon monoxide enters the room at a rate r, and that the well-circulated mixture of air and carbon monoxide is allowed to leave the room at the same rate.

   (a) Assuming that the initial concentration is c_0, find an expression for the concentration $c(t)$ of carbon monoxide in the room at any time.

   (b) Is there a limiting concentration as $t \to \infty$? And if yes, what is its value?

   (c) Prolonged exposure to carbon monoxide concentrations as low as 0.00012 is harmful to the human body. Cigarette smoke typically contains 4% carbon monoxide. For a room of $120ft^3$, initially free of carbon monoxide, with cigarette smoke entering at 0.1 ft^3/min (with well-mixed air removed at the same rate), find the time t at which this concentration is reached.

2. Let

$$Lu = -\frac{d^2u}{dx^2} \quad x \in (0,l), \quad B_1 u = u(0) - u(l), \quad B_2 u = \frac{du}{dx}(0) - \frac{du}{dx}(l).$$

   (a) Is the associated *eigenvalue* problem of Sturm-Liouville type or not? Is the associated *boundary value* problem self-adjoint, formally self-adjoint, or neither? Answer without calculation, but provide a clear statement of your reasoning.

   (b) Construct the Green's function G with the same boundary operators and the differential operator $L - \mu$ where μ is a complex parameter. To do so, simplify the algebra by putting

$$G(x,\xi;\mu) = \begin{cases} A\cos\sqrt{\mu}(x-\xi) - B\sin\sqrt{\mu}(x-\xi) & 0 < x < \xi < l \\ A\cos\sqrt{\mu}(x-\xi) + B\sin\sqrt{\mu}(x-\xi) & 0 < \xi < x < l, \end{cases}$$

   but explain why you may assume this form, since it contains only two constants (A and B) instead of the usual four. Find A and B, and show that your expression for G can be written

$$G(x_<, x_>; \mu) = \frac{-\cos\sqrt{\mu}(x_< - x_> + \frac{l}{2})}{2\sqrt{\mu}\sin\frac{\sqrt{\mu}l}{2}}$$

   where $x_< = \min(x,\xi)$ and $x_> = \max(x,\xi)$.
   Note: You may find these identities useful:
   $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$,
   $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$.

(c) Consider $G(x, \xi; \mu)$ as a function of the complex parameter μ, and find the location of its poles and the residues there. Use the result that

$$\frac{1}{2\pi i} \int_{C_\infty} G(x, \xi; \mu)\, d\mu = -\delta(x - \xi) = -\sum_{n \in S} \phi_n(x)\phi_n(\xi) \qquad (1)$$

and evaluate the integral on the left of this expression, where C_∞ is a circle with center $\mu = 0$ and radius R in the limit $R \to \infty$, to find

i. The eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of $\mathcal{L} = (L, B_i)$. Take care to interpret the sum in (1) correctly, and note you can check your answer by writing down the spectrum directly.

ii. Use the spectral representation of $\delta(x-\xi)$ in (1) and form $\int_0^l f(\xi)\delta(x-\xi)\, d\xi$ to find the eigenfunction expansion of an arbitrary l-periodic function f. What well-known series expansion is this?

3. Consider the eigenvalue problem

$$\frac{d^2u}{dx^2} + \lambda u = 0, \quad 0 < x < 1,$$

subject to the homogeneous boundary conditions

$$\frac{du(0)}{dx} + \lambda u(0) = 0, \quad u(1) = 0$$

where λ is the eigenvalue parameter, which appears explicitly in the boundary condition as well as in the ODE.

(a) Determine the eigenvalues (graphical solutions are acceptable) and corresponding eigenfunctions for this problem.

(b) If $u_m(x)$ and $u_n(x)$ are two distinct eigenfunctions, show that

$$\int_0^1 u_m(x)u_n(x)dx \neq \delta_{mn}, \quad m \neq n,$$

where δ_{mn} is the Kronecker delta function.

(c) Write down and consider the associated inhomogeneous *boundary value* problem, where now λ is a fixed parameter that is not an eigenvalue. Either by calculation or a brief but clear statement of your reasoning, determine whether the problem is self-adjoint, formally self-adjoint, or neither.

4. Consider the channel region defined by $D = \{(x, y)| \quad |x| < \infty, \quad 0 < y < 1\}$.

(a) Find the Green's function satisfying

$$\nabla^2 G + 2ikM\frac{\partial G}{\partial x} + k^2 G = \delta(x - x')\delta(y - y'), \quad (x, y) \in D, \quad (x', y') \in D$$

$$G = 0, \quad y = 0, 1, \quad |x| < \infty$$

where $0 < M < 1$. This function should represent an outgoing wave as $|x| \to \infty$.

(b) Solve the boundary value problem

$$\nabla^2 u + 2\frac{\partial u}{\partial x} + k^2 u = 0, \quad (x, y) \in D$$

$$u(x, 0) = f(x), \quad |x| < 1, \qquad u(x, 0) = 0, \quad |x| > 1.$$

5. Let D be the interior of the unit circle.

(a) Find the Green's function G satisfying

$$G_t = \nabla^2 G + \delta(x - x')\delta(y - y')\delta(t - t'), \quad (x, y) \in D, \quad (x', y') \in D \quad t > t'$$

$$G(x, y, t) = 0, \qquad t < t'$$

$$G(x, y, t) = 0, \qquad x^2 + y^2 = 1.$$

(b) Solve the initial boundary value problem

$$u_t = \nabla^2 u + p(t)\delta(r - r_0), \quad (x, y) \in D$$

$$u(x, y, 0) = 0, \qquad (x, y) \in D$$

$$u(x, y, t) = 0 \qquad x^2 + y^2 = 1$$

where $0 < r_0 < 1$ and $p(t)$ is a smooth function on $t > 0$ and identically zero for $t < 0$. The source in this equation is a model of a wire-ring heater.

6. Consider the initial boundary value problem for the one-dimensional wave equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2}, \quad -\infty < x < L, \quad t > 0$$

$$v = f(x), \quad \frac{\partial v}{\partial t} = 0, \quad |x| < \infty, \quad t = 0$$

$$\frac{\partial v}{\partial t} + r\frac{\partial v}{\partial x} = 0, \quad x = L, \quad t > 0$$

where $r > 0$ and $L > 1$. Assume the support of $f(x)$ is the interval $|x| < 1$, i.e., $f(x) = 0$ for $|x| > 1$. Find the solution to this problem by any means available to you. What happens to the solution when $r = 1$?