

Name: _____

Question:	1	2	3	4	5	6	Total
Points:	15	20	15	20	15	15	100
Score:							

Instructions: Show all work and justify all steps of each argument you make. Points may be deducted if either is missing or inadequate. Note that there are questions on the back. You have $2\frac{1}{2}$ hours for this exam.

1. (15 points) Let A be the matrix

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & -4 \\ 1 & 3 & 1 \\ 1 & 6 & 2 \end{bmatrix}$$

- (a) Let $W = \text{Col } A$. Use the Gram-Schmidt process to compute an orthogonal basis for W . You do not need to normalize the vectors.
- (b) Find a *unit vector* in W^\perp , i.e. find a unit vector \mathbf{v} that is orthogonal to every vector in W .

2. (20 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{bmatrix}.$$

- (a) Find the characteristic polynomial and all the eigenvalues of A .
- (b) Find the eigenvector of A corresponding to the eigenvalue with algebraic multiplicity equal to one.
- (c) Is this matrix diagonalizable? Why or why not? Explain your reasoning, supported by calculations as necessary.
- (d) What is the *rank* of A ?

3. (15 points) Consider the quadratic form

$$q(\mathbf{x}) = 4x_1^2 + 2\sqrt{3}x_1x_2 + 2x_2^2.$$

- (a) Write down the matrix corresponding to this form.
- (b) Find the change of variables $\mathbf{x} = P\mathbf{y}$ that diagonalizes the quadratic form and rewrite $q(\mathbf{x})$ as a function of the components of \mathbf{y} .
- (c) Is this quadratic form positive definite, negative definite, or sign-indefinite? Why?

4. (20 points) Let V be the vector space consisting of continuous functions $f(x)$ defined on the interval $0 \leq x \leq 1$ satisfying $f(0) = f(1) = 0$.

(a) What is the definition of a *subspace*? Let V_n be the set of all functions $f(x)$ with domain $0 \leq x \leq 1$ that can be written as

$$f(x) = \sum_{k=1}^n c_k \sin k\pi x.$$

Is V_n a subspace of V ?

(b) Verify that $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 (1-x^2)f(x)g(x)dx$ is an inner product for the vector space V ; that is, write down all the conditions defining an inner product and check they are true in this case.

(c) Find the length $\|\mathbf{f}\|$ of the vector \mathbf{f} defined by $f(x) = x$ in this inner product space.

5. (15 points) Let $\mathbf{u} = (4, 4, 10)^T$, $\mathbf{w} = (1, 2, 3)^T$ and $W = \text{span}\{\mathbf{w}\}$.

(a) Compute the projection of \mathbf{u} onto W .

(b) What is the distance between \mathbf{u} and the subspace W .

6. (15 points) Consider the matrix V whose three columns are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}, \text{ and } \mathbf{v}_3 = \begin{pmatrix} 6 \\ 6 \\ 8 \end{pmatrix}$$

(a) Are the three vectors linearly independent?

(b) Let $V = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$, i.e. the matrix whose columns are the three given vectors. Find the determinant of V .

(c) Find the general solution to

$$V\mathbf{x} = \begin{pmatrix} 11 \\ 13 \\ 18 \end{pmatrix}.$$