Math 222 FINAL EXAM, December 14, 2007

Read each problem carefully. Show all your work for each problem. No Calculators!

1. (a) (10) Solve the initial value problem \( X = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} X, \ X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

(b) (10) Find all eigenvalues and eigenfunctions of the boundary value problem: \( y'' + \lambda y = 0, \ y'(0) = 0, \ y'(L) = 0 \)

2. (a) (6) Sketch the odd periodic extension of period 2 of \( f(x) \):
\[
f(x) = \begin{cases} 
0, & 0 \leq x < 1/2 \\
1/2 \leq x < 1 
\end{cases}
\]
(Sketch over the interval \([-3,3]\))

(b) (10) Find the Fourier Sine series of \( x \) with period 2.

3. Find the inverse Laplace Transform of the following functions:
(a) (8) \( F(s) = \frac{2s+1}{s^2+2s+5} \)

(b) (8) \( G(s) = \frac{(1-s)e^{-2s}}{(s+1)^2} \)

4. (a) (8) Find the solution \( y(t) \) of the initial value problem \( y'' + y = \delta(t - 2\pi) + \alpha \cos(3t), \ y(0) = 0, \ y'(0) = 0 \)

(b) (4) Calculate \( y'(\frac{2\pi}{3}) \)

(c) (4) Find the value of \( \alpha \) such that \( y'(\frac{7\pi}{2}) = 0. \)

5. State if the following IVP's are linear or nonlinear, and solve the problems:
(a) (8) \( ty' + 2y = \frac{\cos(t)}{t}, \ t > 0, \ y(\pi) = 0 \)

(b) (8) \( y' = \frac{2x}{y+x^2y}, \ y(0) = -2 \)

6. (a) (8) Determine the appropriate form of a particular solution \( y_p \) for the differential equation: \( y'' + 6y' + 13y = e^{-3x}\sin(2x) + x^2\cos(3x) \).
( DO NOT evaluate the constants in \( y_p \).)

(b) (8) Given that \( y_1 = x^2 \) is a solution of the differential equation \( x^2 y'' - 3xy' + 4y = 0, \ x > 0, \)
use reduction of order to find the second linearly independent solution.