Math 222 Final Exam, May 8, 2015

Read each problem carefully. Show all your work for each problem. No calculators!

1. (12) Express the general solution of the system $\boldsymbol{x}' = \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix} \boldsymbol{x}$ in terms of real-valued functions. Then find the solution with initial condition $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

2. (a) (6) Sketch the odd and the even periodic extension of period 4 of $f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & 1 \le x < 2, \end{cases}$ over the interval $[-6, 6]$.

   (b) (8) Sketch the graph of the function $f(x) = \begin{cases} 1 + x, & -1 \le x < 0, \\ 1 - x, & 0 \le x < 1, \end{cases}$ with $f(x+2) = f(x)$ for three periods, then find its Fourier series.

3. (a) (7) For the boundary value problem $y'' + 9y = \cos x$, $y'(0) = 0$, $y'(\pi) = 0$, either find the solution or show that there is no solution.

   (b) (7) Find the eigenvalues and eigenfunctions of the boundary value problem $y'' + \lambda y = 0$, $y(0) = 0$, $\frac{dy}{dx}(L) = 0$. You can assume that there are no negative eigenvalues.

4. (12) Use Laplace transforms to find the solution of the initial value problem

$$y'' + 2y' + 10y = \begin{cases} 0 & 0 \le t < 2, \\ 1 & t \ge 2, \end{cases} \qquad y(0) = 0, \quad y'(0) = 0.$$

   How does the solution behave as $t \to \infty$?

5. Find the general solution in terms of real-valued functions of

$$\text{(a) (6)} \quad y'' + 2y' + \frac{5}{4}y = 0\,, \quad \text{(b) (6)} \quad x^2y'' - 3xy' + 4y = 0\,.$$

6. Use the method of undetermined coefficients to:
   (a) (6) Solve the initial value problem $y'' + 4y = 2\sin 2t$, $y(0) = 1$, $y'(0) = 3$.
   (b) (6) Find the form of the general solution of $y'' + 3y' + 2y = e^{-t} + t^2 + 2t$. You do **not** need to evaluate the coefficients for this part of the question.

7. (12) For the ODE

$$x^2y'' - x(x+2)y' + (x+2)y = 2x^3\,,$$

   verify that $y_1 = x$ and $y_2 = xe^x$ are solutions of the homogeneous equation, then find the general solution by using variation of parameters.

8. Find the general solution of

$$\text{(a) (6)} \quad \frac{dy}{dx} = xy^2(1+x^2)^{-1/2}, \quad \text{(b) (6)} \quad t\frac{dy}{dt} + (t+2)y = 1\,.$$