Math 222 Exam 1, February 11, 2015

Read each problem carefully. Show all your work for each problem. No calculators!

1. (a) (8) Given the differential equation $y' = -y(2-y)$, sketch the direction field for $y \geq 0$ and $y < 0$. Describe the behavior of solutions as $t \to \infty$ and how this depends on the initial value of y at $t = 0$.

   (b) (8) Verify that $y_1 = t^2$ and $y_2 = t^2 \ln t$ are solutions of the ODE $t^2 y'' - 3ty' + 4y = 0$.

2. (16) Determine the order of each differential equation and whether it is linear or nonlinear

$$(a)\;\; (1-x^2)y'' - 2xy' + 2y = 0, \qquad (b)\;\; \frac{d^3y}{dt^3} + (t\cos t)y^2 = t.$$

3. (a) (9) Find the solution of the initial value problem

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \quad y(\pi) = 2\,.$$

   Does the solution tend to a limit as $t \to \infty$, and if so what is its value?

   (b) (9) Find the solution of the initial value problem, and give y in terms of x explicitly

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{(1+x^2)}}, \quad y(0) = 1\,.$$

4. A tank has capacity 6 L and initially contains 11 mg of salt dissolved in 3 L of water. A solution containing 1 mg/L of salt enters the tank at a rate of 3 L/hour and the well-stirred mixture leaves the tank at a rate of 2 L/hour.

   (a) (4) Find the time when the tank is full.

   (b) (14) Find the amount of salt (in milligrams) in the tank at any time before the tank is full. What is the amount of salt (in mg) in the tank when it is full and what is the concentration of the mixture then (in mg/L)?

5. (16) The initial value problem for Newton's law of cooling is

$$\frac{dT}{dt} = -\kappa(T - T_1)\,, \quad T(0) = T_0\,,$$

   where $\kappa > 0$ is constant and T_1 is the constant ambient temperature. Solve the initial value problem to find how the temperature of the object evolves for $t > 0$.

6. (a) (8) Find the general solution of the differential equation $2y'' - 3y' + y = 0$.

   (b) (8) Solve the initial value problem $y'' + y' - 6y = 0$, $y(0) = 1$, $y'(0) = \beta$. For what value of β does the solution tend to zero as $t \to \infty$?