Ph.D. Qualifying Exam Linear Algebra and Applied Statistics and Probability - May 22, 2003
1:00 p.m. to 4:00 p.m.

1. (a) Consider an $n \times n$ Hermitian matrix A with eigenvalues

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n.$$

   What can we say about the eigenvalues of B,

$$\beta_1 \geq \beta_2 \geq \ldots \geq \beta_{n-1},$$

   where B is the $(n-1) \times (n-1)$ matrix formed by deleting the last row and last column of A?

   (b) For complex $n \times n$ matrices A and C related by the unitary transformation $C = U^*AU$, where U is an $n \times n$ matrix satisfying $U^*U = I$ and I is the identity matrix, show that A and C have the same eigenvalues. For the case when A is a 3×3 matrix, use a suitable unitary matrix U to show that the result of part (a) holds when the second row and second column of A are deleted to form B.

2. Let V be the real vector space of functions spanned by the set of real valued functions $\{e^x, xe^x, x^2e^x\}$, and let T be the linear operator on V defined by $T(f) = f\ '$, the derivative of f. Find the Jordan canonical form of T, and find a Jordan canonical basis.

3. Let A be an $n \times n$ complex matrix with characteristic polynonmial

$$f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

   (a) Show that

$$dim(span(\{I, A, A^2, \ldots, A^n\})) \leq n.$$

   (b) When A is a normal matrix, i.e., one having a complete set of mutually orthogonal eigenvectors, show that

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0.$$

4. State and prove the Weak Law of Large Numbers.

5. Let $X_1, X_2, \ldots, X_n$, $n \geq 2$, be independent and identically distributed with density

$$f(x, \underset{\sim}{\theta}) = \frac{1}{\sigma} \exp\left\{\frac{-(x-\mu)}{\sigma}\right\} I_{[\mu,\infty)}(x),$$

where $\underset{\sim}{\theta} = (\mu, \sigma^2), -\infty < \mu < \infty, \sigma^2 > 0$.

(a) Find maximum likelihood estimates of μ and σ^2.

(b) Find the maximum likelihood estimate of $P_{\underset{\sim}{\theta}}[X_1 \geq t]$ for $t > \mu$.

6. Let $X_1, X_2, \ldots, X_n$ denote a random sample from

$$f(x, \theta) = \left(\frac{1}{\theta}\right) x^{\frac{(1-\theta)}{\theta}} I_{(0,1)}(x).$$

Test H$_o$: $\theta \leq 1$ versus H$_1$: $\theta > 1$ For a sample of size n, find a uniformly most powerful size-α test if such exists.