Doctoral Qualifying Exam
Department of Mathematical Sciences
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Statistics Part B: Real Analysis and Statistical Inference — August 2014

The first three questions are about Real Analysis and the next three questions are about Statistical Inference.

1. (a) State the definition of a measure.

   (b) Prove that if $A \subset B$, then $\mu(A) \leq \mu(B)$.

   (c) Prove that if $A_1 \subset A_2 \subset A_3 \subset \ldots$, then

$$\lim_{j \to \infty} \mu(A_j) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right).$$

   (d) Prove that if $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\mu(A_1) < \infty$, then

$$\lim_{j \to \infty} \mu(A_j) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right).$$

2. The Hardy-Littlewood-Sobolev inequality reads:

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} g(y)\, dx\, dy \right| \leq C \|f\|_p \|g\|_q,$$

   where $p, q > 1$, $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $0 < \lambda < n$ with $\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{q} = 2$, and $C > 0$ depending only on p, q and n.

   (a) Show that this inequality *cannot* hold for any $0 < \lambda < n$ such that

$$\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{q} \neq 2.$$

   (b) If $\lambda = n - 2$ and $p = q$, for which values of n and p does the Hardy-Littlewood-Sobolev inequality hold?

   (c) Is it possible to choose $p = q = 2$ in the Hardy-Littlewood-Sobolev inequality?

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} \sqrt{1 - |x|^2}, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

   (a) Sketch the graph of this function. Does this function belong to any of these classes (justify your answer): $C(\mathbb{R}^2)$, $C^1(\mathbb{R}^2)$, $C^\infty(\mathbb{R}^2)$, $C_c(\mathbb{R}^2)$, $C_c^1(\mathbb{R}^2)$, $C_c^\infty(\mathbb{R}^2)$?

   (b) (**extra credit**) Prove that this function belongs to $W^{1,p}(\mathbb{R}^2)$ for any $p \in [1, 2)$.

4. Let $X_1, \ldots, X_n$ be independent identically distributed $\mathrm{N}(0, \theta)$ where $0 < \theta < \infty$.

   (a) Find a sufficient statistic for θ.

   (b) Find an unbiased estimator of θ based on the above sufficient statistic and find the variance of the estimator.

5. Let $X_1, \ldots, X_n$ be an independent identically distributed sample from the Cauchy distribution with probability density function given by

$$f(x; \theta) = \frac{1}{\pi[1 + (x - \theta)^2]}, \qquad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

   (a) Find the Cramer-Rao lower bound for an unbiased estimator of θ.

   (b) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$ if $\hat{\theta}$ is the maximum likelihood estimator of θ?

6. Let $X_1, \ldots X_n$ be a random sample from a gamma distribution with parameters $\alpha = 3$ and $\beta = \theta$. Let $H_0 : \theta = 2$ versus $H_1 : \theta > 2$.

   (a) Show that there exists a uniformly most powerful test for H_0 against H_1, determine the statistic Y upon which the test may be based, and indicate the nature of the best critical region.

   (b) Find the probability density function of the statistic Y in this problem part (a). If we want a significance level of 0.05, write an equation which can be used to determine the critical region. Let $\gamma(\theta), \theta \geq 2$, be the power function of the test. Express the power function as an integral.