Doctoral Qualifying Exam
Department of Mathematical Sciences
New Jersey Institute of Technology

Applied Math Part B: Real and Complex Analysis — August 2014

The first three questions are about Real Analysis and the next three questions are about Complex Analysis.

1. (a) State the definition of a measure.

   (b) Prove that if $A \subset B$, then $\mu(A) \le \mu(B)$.

   (c) Prove that if $A_1 \subset A_2 \subset A_3 \subset \ldots$, then

$$\lim_{j\to\infty} \mu(A_j) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right).$$

   (d) Prove that if $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\mu(A_1) < \infty$, then

$$\lim_{j\to\infty} \mu(A_j) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right).$$

2. The Hardy-Littlewood-Sobolev inequality reads:

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} g(y)\, dx\, dy \right| \le C\|f\|_p \|g\|_q,$$

   where $p, q > 1$, $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $0 < \lambda < n$ with $\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{q} = 2$, and $C > 0$ depending only on p, q and n.

   (a) Show that this inequality *cannot* hold for any $0 < \lambda < n$ such that

$$\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{q} \neq 2.$$

   (b) If $\lambda = n - 2$ and $p = q$, for which values of n and p does the Hardy-Littlewood-Sobolev inequality hold?

   (c) Is it possible to choose $p = q = 2$ in the Hardy-Littlewood-Sobolev inequality?

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} \sqrt{1 - |x|^2}, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

   (a) Sketch the graph of this function. Does this function belong to any of these classes (justify your answer): $C(\mathbb{R}^2)$, $C^1(\mathbb{R}^2)$, $C^\infty(\mathbb{R}^2)$, $C_c(\mathbb{R}^2)$, $C_c^1(\mathbb{R}^2)$, $C_c^\infty(\mathbb{R}^2)$?

   (b) (**extra credit**) Prove that this function belongs to $W^{1,p}(\mathbb{R}^2)$ for any $p \in [1, 2)$.

4. **Series and singularities**

Caterogorize all zeros and singularities of the following function (make sure to analyze $z = \infty$ as well), and find two dominant terms in the series expansion of the function around $z_o = 1$; assume the branch $-\pi \leq \arg(z) < \pi$ for $z^{1/2}$

$$f(z) = \frac{\sin^2(\pi z)}{z^{1/2} - 1}$$

5. **Complex integration and residues**

Calculate the following integrals, carefully explaining all steps:

$$\textbf{(a)} \int_{-\infty}^{\infty} \frac{x \ \sin x \ dx}{x^2 - 2x + 2} \qquad \textbf{(b)} \int_{|z|=1/2} \frac{z \ dz}{\cos(1/z)}$$

In (b), the contour of integration is a circle of radius $1/2$ centered at the origin; make the transformation $w = 1/z$ before calculating this integral (don't forget to map the contour of integration onto the w-plane).

6. **Corollaries of the Cauchy Integral Formula**

Suppose $f(z)$ is analytic in domain D and that $|f(z)| \leq M$ in D. Prove that for all points in D, we have $|f'(z)| \leq \frac{M}{d}$, where d is the (shortest) distance from z to the boundary of domain D (hint: use the Cauchy Integral Formula).