Ph.D. Prelim: Exam. C
Probability Theory

August 27, 2014

1. This question has two parts, **(i)** and **(ii)** below.

(i) For a sequence $\{X_n, n \geq 1\}$ of random variables, state the condition that guarantees the statement: X_n converges to 0 in probability implies that the convergence holds in L^p. Prove the statement assuming the condition.

(ii) Prove that X_n converges to 0 in probability if and only if

$$E\left(\frac{|X_n|}{1+|X_n|}\right) \to 0.$$

2. This question has two parts, **(i)** and **(ii)** below.

(i) Let $\{\theta_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ be a double array of complex numbers satisfying

$$\begin{aligned}
\max_{1\leq j\leq k_n} |\theta_{nj}| &\to 0, \qquad \text{as } n \to \infty, \\
\sum_{j=1}^{k_n} |\theta_{nj}| &\leq M < \infty, \text{ where } M \text{ does not depend on } n, \\
\sum_{j=1}^{k_n} \theta_{nj} &\to \theta, \text{ as } n \to \infty, \text{ where } \theta \text{ is a finite complex number.}
\end{aligned}$$

Show that

$$\prod_{j=1}^{k_n} (1+\theta_{nj}) \to e^{\theta}.$$

(ii) For each $n \geq 1$, let $X_{nj}, j = 1, \ldots, n$ be independent random variables. Let $S_n = \sum_{j=1}^{n} X_{nj}$. Suppose that

$$P\{X_{nj} = 1\} = p_{nj} = 1 - P\{X_{nj} = 0\},$$

and that

$$\sum_{j=1}^{n} p_{nj} \to \lambda \text{ as } n \to \infty, \qquad \max_{1\leq j\leq n} p_{nj} \to 0 \text{ as } n \to \infty.$$

Show that S_n converges in distribution to a Poisson random variable with parameter λ.

3. This question has two parts, **(i)** and **(ii)** below.

(i) State and prove Kolmogorov's strong law of large numbers.

(ii) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables and let $E(|X_i|) < \infty$. Show that $X_n/n \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. Hence also prove that $\max_{1\leq j\leq n} |X_j|/n$ converges to zero almost everywhere as $n \to \infty$.