Doctoral Qualifying Exam

Department of Mathematical Sciences
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Applied Math Part C: Linear Algebra and Numerical Methods — August 2014

The first three questions are about Linear Algebra and the next three questions are about Numerical Methods.

1. Consider the linear system of equations

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_1 + 3x_2 + x_3 &= 1 \\ x_2 + 2x_3 &= 3 \end{aligned} \tag{1}$$

   (a) Does this system have a unique solution? If so, find it.

   (b) Add a fourth equation

$$a_1x_1 + a_2x_2 + a_3x_3 = b$$

   Will the new system always have a unique solution? If so, prove it. If not, find the conditions on the a_i $(i = 1, 2, 3)$ and b so that the new system does have a unique solution.

2. Suppose that v is a nonzero column vector in $\mathbb{C}^n$ $(n > 1)$ and the matrix $A = vv^H/(v^Hv)$.

   (a) What are the eigenvalues of A? Explain.

   (b) Is the matrix $I + A$ (I is the $n \times n$ identity matrix) diagonalizable? Explain.

   (c) Find the determinant of $I + A$.

   (d) What is A^{2014}? Explain.

3. (a) Suppose that $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ are orthonormal bases for $\mathbb{R}^n$. Construct the matrix A that transforms each v_j into u_j to give $Av_1 = u_1, \cdots, Av_n = u_n$.

   (b) Find the SVD of the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

4. (a) Apply the backward Euler method to the problem $Y' = \lambda Y$ for $x > 0$ with $Y(0) = 1$ and λ an arbitrary real constant. Let $y_h(x_n)$ be the numerical approximation to the true solution evaluated at x_n with a step size h. Show that the error is

$$Y(x_n) - y_h(x_n) = -\frac{\lambda^2 x_n e^{\lambda x_n}}{2} h + O(h^2).$$

   (b) Apply the trapezoidal method to $Y' = \lambda Y$ for $x > 0$ with $Y(0) = 1$ and λ an arbitrary real constant. Show first that

$$y_h(x_n) = \left(\frac{1 + \lambda h/2}{1 - \lambda h/2}\right)^n,$$

   and then show that

$$Y(x_n) - y_h(x_n) = -\frac{\lambda^3 x_n e^{\lambda x_n}}{12} h^2 + O(h^4).$$

5. Show that the following iteration for root finding is a second-order method:

$$x_{n+1} = x_n - \frac{f(x_n)}{D(x_n)}, \quad D(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}, \quad n \geq 0.$$

6. Consider the following two-step method for solving the initial value problem $y' = f(x, y)$, $y(0) = y_0$:

$$y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{h}{4}\left[4y'_{n+1} - y'_n + 3y'_{n-1}\right], \quad n \geq 1$$

with $y'_n \equiv f(x_n, y_n)$ and h is the step size. Show that it is second-order, and find the leading term in the truncation error. Discuss the stability of this two-step method.