Department of Mathematical Sciences
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Part B: Real and Complex Analysis — Doctoral Qualifying Exam, August 2013

The first three questions are about Real Analysis and the next three questions are about Complex Analysis.

1. Prove the Schwartz inequality

$$|(f,g)| \leq ||f|| \cdot ||g||$$

Hint: consider the expression $||f+tg||^2$, where t is a scalar, this expression is a quadratic polynomial of t, and then find the value of its maximum.

Deduce that

$$\sum a_n b_n \leq \left(\sum a_n^2\right)^{1/2} \left(\sum b_n^2\right)^{1/2}$$

2. Show that

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} f(x,t)dx\right] = f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)dx$$

provided f and $\partial f/\partial t$ are continuous on $[a_0, b_0] \times B$ where B is open, that $a_0 \leq a(t) \leq b_0$ and $a_0 \leq b(t) \leq b_0$ for every $t \in B$, and that the functions a, b are of class $C^{(1)}$. (Hint: let $G(x,t) = \int_{a_0}^{x} f(s,t)ds$ so that $\partial G/\partial x = f$. Calculate the derivative of $(G(b(t),t) - G(a(t),t))$.

3. Let us consider a continuous functions $f(x)$ as well as its derivative, where $f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$ and $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos ny dy$, $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny dy$. Let us also consider the partial sum $S_N(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx)$.

   - Consider the function $K_N(\theta) = 1 + 2\sum_{n=1}^{N}\cos n\theta$. Show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)d\theta = 1$.
   - Using $\cos x = (e^{ix} + e^{-ix})/2$, show that $K_N(\theta) = \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$.
   - Show that the functions $X_n = \sin(n+\frac{1}{2})\theta$ for $\theta \in (-\pi, 0)$ and $\theta \in (0,\pi)$ are orthogonal.
   - Use the Bessel's inequality

     $$\sum_{n=0}^{\infty} \frac{(g_+, X_n)}{||X_n||^2} \leq ||g_+||^2$$

     to show that $(g_+, X_n) \to 0$ as $n \to \infty$, where in this case $f(x)$ is a piecewise continuous functions (and its derivative) such that $f(x) = g_+(x)$ for $x \in (0,\pi)$ and $f(x) = g_-(x)$ for $x \in (-\pi, 0)$.
   - Do we have the sane result for (g_-, X_n)?

4. Let f be a complex-valued analytic function on a domain D in the complex plane $\mathbb{C}$.

(a) Use the Cauchy–Riemann equations to prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\left|f'(z)\right|^2$$

on D.

(b) Prove using (a) that $|f|$ is constant on D iff f is constant on D.

5. Consider the function

$$\varphi(z) := \frac{z}{z^4 - 16}.$$

(a) Find and classify all singularities of φ, and compute the residue at $z = 2$.

(b) Find the Laurent series expansion of φ for $|z| > 2$.

(c) φ has a single zero in $B_2(0) := \{z \in \mathbb{C} : |z| < 2\}$. Given any positive integer n, construct an entire function having n distinct zeros in $B_2(0)$.

(d) What can you say about an analytic function having infinitely many distinct zeros in $B_2(0)$? Explain your answer.

6. Let Φ be a function analytic on a domain containing the closure of the upper half-plane $\mathbb{H}_+ := \{z = x + iy \in C : y > 0\}$, and suppose that there exists a positive constant M such that $|\Phi(z)| \leq M/|z|$ for all sufficiently large values of $|z|$. Use the calculus of residues to give a detailed proof that

$$\Phi(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Phi(x)dx}{x - z} \qquad \forall z \in \mathbb{H}_+.$$

(Hint: Use expanding upper semicircular contours)