Department of Mathematical Sciences
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Part C: Linear Algebra and Numerical Methods — Doctoral Qualifying Exam, August 2013

The first three questions are about Linear Algebra and the next three questions are about Numerical Methods.

1. Consider
$$A = \begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & 1 \end{pmatrix}.$$
(a) Find the four fundamental subspaces (row space, column space, nullspace and left nullspace) of A.
(b) Compute the singular value decomposition of A.
(c) Find the minimum-length solution to $Ax = b$ if
$$b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

2. Suppose we have that
$$U_1 D_1 U_1^* = U_2 D_2 U_2^* + U_3 D_3 U_3^*$$
where $n \times n$ matrix U_j is unitary for each j and $n \times n$ matrix $D_j = \text{diag}\{\mathbf{d_j}\}$ is diagonal for each j and satisfies $d_{j1} > d_{j2} > \cdots > d_{jn}$.
Is it true that $D_1 = D_2 + D_3$? If so, prove the statement. If not, provide a *specific* counterexample. Note that U^* is the Hermitian conjugate of U.

3. Linear transformation T is referred to as *normal* if $TT^* = T^*T$.
(a) If W is the kernel (nullspace) of normal transformation T, show that $T^*(W) \subset W$.
(b) Let $T = \frac{d}{dx} : V \to V$ be the differentiation operator acting on space V of degree-3 polynomials with basis $\{x^3 - x^2, x^2 - x, x - 1, x + 1\}$. Find the matrices associated with operators T and T^*, and show whether or not T is normal.
(c) For the operator T in part (b), show explicitly whether or not the property in part (a) holds.

4. Assume that $f : \mathbf{R} \to \mathbf{R}$ is a smooth function, satisfies
$$f'(x) > 0, \quad f''(x) > 0, \qquad \text{for all } x \in \mathbf{R},$$
and has a root x^*. Prove that the Newton's method converges to x^* for any initial guess $x_0 \in \mathbf{R}$.

5. Let N be a positive integer, $h = 1/N$, $x_n = nh, n = 0, 1, \ldots, N$. For $n = 1, 2, \ldots N$, define $\psi_n \in L^2([0,1])$ by
$$\psi_n(x) = \begin{cases} 1 & , \quad x \in (x_{n-1}, x_n) \\ 0 & , \quad \text{else} \end{cases} \tag{1}$$
Set $S_h = span\{\psi_1, \psi_2, \ldots \psi_n\}$, and let $P_h : L^2([0,1]) \to S_h$ denote the orthogonal L^2 orthogonal projection.
(a) Find an explicit expression for the coefficients c_n in the expansion $P_h f = \sum_{n=1}^N c_n \psi_n$.
(b) Prove that $\|f - P_h f\|_{L^\infty([0,1])} \le Ch\|f'\|_{L^\infty([0,1])}$, $f \in C^1([0,1])$, with C a constant independent of f and h.

6. Consider the initial value problem

$$y' = f(t, y), \; y(0) = y_0.$$

Derive an explicit, two-stage second-order Runge-Kutta method for the approximate solution of this problem of the form

$$y_{n+1} = y_n + h[\alpha_1 f(t_n, y_n) + \alpha_2 f(t_n + \theta h, y_n + k_n)], \; k_n = \beta h f(t_n, y_n).$$

Justify your answer.