Preliminary Exam, Part B: Real and Complex Analysis
Tuesday August 28, 2012

You have three hours for this exam. Show all working in the answer books provided. To facilitate grading, keep your answers to the first three questions (on real analysis) separate from your answers to the last three questions (on complex analysis). Good luck.

1. Compute the following limits and justify the calculation.

(a) $\lim_{n\to\infty} \int_0^1 n \ln\left(1+\frac{x}{n}\right)\ dx$.

(b) $\lim_{n\to\infty} \int_0^\infty n \sin(x/n)[x(1+x^2)]^{-1}\ dx$.

2. Let $C([0,a])$ be the space of continuous functions on $[0,a]$ with the sup norm metric. Consider the integral equation

$$y(x) = 1 + \int_0^x f(t, y(t))\ dt, \quad y(0) = 1.$$

Assume f is defined, continuous and $|f| < M$ on $\mathbb{R}^2$ and that f is Lipschitz continuous in the second variable, i.e., for $L > 0$

$$|f(x,y) - f(x,z)| \leq L|y-z|\,.$$

(a) Show that there is a unique solution in $C([0,a])$ for sufficiently small $a > 0$. (Hint: Define

$$Tu = 1 + \int_0^x f(t, u(t))\ dt$$

and consider the iteration $u^{n+1} = Tu^n$ with $u^0 \in C([0,a])$. You need to use the fact that $C([0,a])$ is complete with respect to the sup norm.)

(b) Show that, in fact, the solution exists for all $a > 0$. (Hint: Let $u^0 = 1$ and show $|u^n(x) - u^{n-1}(x)| \leq L^{n-1}Mx^n/n!$)

3. Assume $f \in L(I)$, where I is a bounded subset $I \subset \mathbb{R}$. The Riemann-Lebesgue lemma states that for each real β,

$$\lim_{\alpha\to\infty} \int_I f(t)\sin(\alpha t + \beta)\ dt = 0.$$

(a) Prove the Riemann Lebesgue lemma for the characteristic function on $[a,b]$ (i.e., $f = 1$ on $[a,b]$ and 0 otherwise).

(b) Prove the lemma for f a step function on I.

(c) Prove the lemma for $f \in L(I)$.

4. For the complex function

$$\varphi(z) = \frac{1}{\sin\left(\frac{\pi}{z}\right)}$$

(a) Describe all singularities of $\varphi(z)$, including any singularity at infinity.

(b) Compute the residue of $\varphi(z)$ at $z = 1$.

(c) Using long division of power series or any other means, find the first *three* nonzero terms in the Laurent series expansion of $\varphi(z)$ for $|z| > 1$. (Hint: recall that $\sin w = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!}$ for all w.)

5. Consider the equation

$$z + e^{-z} = \alpha,$$

where $\alpha > 1$.

(a) By using Rouche's theorem, the principle of the argument, or otherwise, show that this equation has a unique solution in the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ by considering a sufficiently large closed semicircular curve in H.

(b) Use the result of part (a) to show that the solution of this equation in H is real.

6. Let a real-valued function of the real variable x have the power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where all of the coefficients a_n are real and the radius of convergence of the series is $R = \lim_n |a_n/a_{n+1}| = 1$.

(a) Explain why f has a unique analytic extension F to the open unit disk in the complex plane $\mathbb{C}$.

(b) Since $R = 1$, we know that F cannot be extended to an analytic function on $\{z \in \mathbb{C} : |z| < 1 + \epsilon\}$ for any $\epsilon > 0$. Therefore, F must have a singularity on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. *Show that this singularity need not be a pole* by finding an example where $|F|$ is bounded in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and explain why this precludes the existence of a pole on S^1. (Notice that $\sum_{n=0}^{\infty} z^n = (1 - z)^{-1}$ is *not* such an example since its modulus increases without bound as $z \to 1$ from the interior of the unit disk).