Preliminary Exam, Part C: Linear Algebra and Numerical Methods
Monday August 27, 2012

You have three hours for this exam. Show all working in the answer books provided. To facilitate grading, keep your answers to the first three questions (on linear algebra) separate from your answers to the last three questions (on numerical methods). Good luck.

1. Let X be the real vector space of real polynomials p of degree less than or equal two, and $L : X \to X$ be the linear differential operator defined as $L = D^2 + I$, where $D = d/dx$ and I is the identity

(a) Show that the *kernel* (or *null space*) of L, $\ker L := \{p \in X : L(p) = 0\}$, is equal to $\{0\}$ and explain why this implies that L is invertible.

(b) Compute the matrix of L with respect to the basis of monomials $\{1, x, x^2\}$

(c) Prove that the inverse of L is $-D^2 + I$.

2. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & 1/4 \end{pmatrix}.$$

Show, either by direct computation and/or invoking the relevant spectral theorems, that for any $x \in \mathbb{R}^3$ the iterates $A^n x$ converge to the eigenspace (or eigenvector) $\Lambda(1) = \{x \in \mathbb{R}^3 : (A - I)\, x = 0\}$ corresponding to the eigenvalue $\lambda = 1$.

3. Let A be a complex Hermitian $n \times n$ matrix. Use the standard inner product $\langle \cdot, \cdot \rangle$, the definition of Hermitian, and the appropriate spectral theorem to prove that:

(a) All the eigenvalues of A are real, and

(b) A is positive-definite (i.e. $\langle Ax, x \rangle > 0$ for all $x \neq 0$ in $\mathbb{C}^n$) if and only if all of the eigenvalues of A are positive.

4. Consider the secant method

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n \geq 1$$

for finding one or more roots of an equation $f(x) = 0$. Show that the error estimate formula for the secant method is given by

$$\alpha - x_{n+1} = -(\alpha - x_{n-1})(\alpha - x_n) \frac{f''(\zeta_n)}{2f'(\xi_n)},$$

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where $f(\alpha) = 0$, ξ_n is between x_{n-1} and x_n, and ζ_n is between the largest and the smallest of x_{n-1}, x_n and α.

5. The goal of minimizing the maximum deviation of a polynomial $q_n(x)$ from a function $f(x)$ on the interval $[a, b]$, i.e. minimizing the maximum error $\max_{x\in[a,b]} |f(x) - q_n(x)|$, is called minimax approximation. Let $q_1(x) = a_0 + a_1 x$ be the linear minimax approximation to $f(x)$ on $[-1, 1]$, where $f \in C^2[-1, 1]$ with $f''(x) > 0$ for $x \in [-1, 1]$.

(a) Show that

$$a_1 = \frac{f(1) - f(-1)}{2}, \quad a_0 = \frac{f(-1) + f(\chi)}{2} - \left(\frac{\chi - 1}{2}\right) \left[\frac{f(1) - f(-1)}{2}\right],$$

where χ is the unique solution of

$$f'(\chi) = \frac{f(1) - f(-1)}{2}.$$

(b) Find the maximum error.

6. For the initial value problem for the ODE $\frac{dy}{dt} = f(t, y)$, $y(0) = y_0$, its numerical solution by Simpson's method is given by

$$y_{n+1} = y_{n-1} + \frac{h}{3} \left[f(t_{n-1}, y_{n-1}) + 4f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right], \quad n \geq 1.$$

Show that the method has a parasitic (or 'spurious') solution by applying Simpson's method to the specific example $\frac{dy}{dt} = \lambda y$, $y(0) = 1$, where λ is real and negative, and finding a numerical solution that does not correspond to the (true) solution of this specific example.