Preliminary Exam in Real and Complex Analysis: Summer 2011

1. Assume $\{\phi_0, \phi_1, \ldots\}$ is a collection of L_2 functions which form an orthonormal set on some interval I. Let $\{c_n\}$ be any sequence of complex numbers such that $\Sigma|c_k|^2$ converges. Show that there exists a function $f \in L_2(I)$ such that $(f, \phi_k) = c_k$ for each $k \geq 0$.

2. Use Fourier series methods to establish and justify the following results.

   (a) For $0 < x < \pi$
   $$\frac{\pi}{4} = \Sigma_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

   (b) For $0 \leq x \leq \pi$
   $$x = \frac{\pi}{2} - \frac{4}{\pi}\Sigma_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

3. Consider $\mathbf{R}$, with $\mathcal{B}$ the Borel σ-algebra and μ the Lebesgue measure.

   (a) Let $a_n \in (0,1)$ be an increasing sequence of points that converges to 1. Define $E_n = [0, a_n]$. Let $f_n(x) = \chi_{E_n}(x)$ where χ is the characteristic function. Does $f_n(x)$ converge almost everywhere, uniformly, in L_p for any $p \geq 1$, in measure or almost uniformly? Show your work.

   (b) Compute the value of the following integral.
   $$\lim_{n\to\infty} \int_0^{\infty} e^{-nx} \frac{1}{\sqrt{x}}\, d\mu.$$

   Be sure to provide justification for all of your work.

4. Integrate using a "keyhole" contour surrounding the branch cut of function x^p, and state conditions on the values of complex parameters p and a required for convergence of this integral:
   $$\int_0^{\infty} \frac{x^p\, dx}{x+a}$$

5. Use Rouché's theorem to show that all roots of the following polynomial are enclosed within a ring $1 < |z| < 2$:
   $$f(z) = z^5 - 5z + 21$$
   Hint: Apply the Rouché's theorem to domains $|z| < 1$ and $|z| < 2$.

   Rouché's theorem: If f, g are analytic inside and on a simple closed contour C and $|f| > |g|$ on C, then f and $f + g$ have the same number of zeros within the domain bounded by C.

6. Laurent Series

(a) Find the Laurent series of the following function converging in the indicated domain:

$$f(z) = \log\left\{\frac{z-1}{z+2}\right\}, \quad |z| > 2$$

Hint: assume the branch cut is the interval $[-2, 1]$, therefore the function is analytic in $|z| > 2$. You can use an appropriate Taylor expansion of logarithmic function to find this series.

(b) Find the negative-power terms in the Laurent series for the function $1/\cos z$ converging in domain $\frac{\pi}{2} < |z| < \frac{3\pi}{2}$, to show that this series has the following form:

$$\frac{1}{\cos z} = 1 - \frac{4}{\pi}\sum_{m=0}^{\infty}\left(\frac{\pi}{2z}\right)^{2m} + \sum_{m=1}^{\infty} C_{2m} z^{2m}$$

Hint: proceed directly from the integral defining coefficients of Laurent series converging in $r < |z - z_0| < R$:

$$C_n = \frac{1}{2\pi}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}, \quad n = 0, \pm 1, \pm 2, \pm 3, ...$$

where C is a simple closed contour within $r < |z - z_0| < R$ and enclosing the inner boundary of this domain.