Preliminary Exam in Linear Algebra and Numerical Methods: Summer 2011

1. Suppose Stephen (S), Michael (M) and Jack (J) are competing in an election. Each week, the candidates lose a percentage of their votes to other candidates according to the following table:

S to M:	5%	S to J:	5%
M to J:	49%	M to S:	1%
J to M:	49%	J to S:	1%

   (a) Describe the week-to-week change in vote distribution as a Markov process.

   (b) Suppose the distribution of votes is identical in weeks seven and eight. What is the final distribution of votes?

2. Show that a diagonalizable matrix satisfies its characteristic equation. Is this also true for noninvertible matrices?

3. Let P and Q be real invertible matrices, and let X be the set of minimizers of $\|P(Ax-y)\|$ (i.e., $X = \{x \in \mathbb{R}^n \,|\, x \text{ minimizes } \|P(Ax-y)\|\}$. Then it can be shown that the vector x^- in X that minimizes $\|Qx\|$ is given by

$$x_- = A^- y := Q^{-1}(PAQ^{-1})^+ Py$$

   where the '+' symbol denotes the pseudoinverse. (Recall that if $x^+ = A^+ y$ then $A^T A x^+ = A^T y$.)

   (a) Suppose now that both P and Q are (real) orthogonal matrices. Show that $(PAQ^T)^+ = QA^+P^T$ by relating solutions to their corresponding normal equations, then use this fact to show that $A^- = A^+$.

   (b) Find the minimum-length least-squares solution to

$$\begin{pmatrix} 2 & 1 \\ 0 & 4 \\ 0 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

   with weighting matrices

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \qquad Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}.$$

4. Consider the fixed point iteration method $x_n = g(x_{n-1})$. Assume that

$$|x_n - x_{n-1}| \le \lambda^{n-1} |x_1 - x_0|$$

   where $\lambda < 1$. Show that this sequence $\{x_n\}$ satisfies the Cauchy criterion. Hint: A sequence $\{x_n\}$ satisfies the Cauchy criterion if given any ϵ, $\exists\, N$ such that $|x_n - x_m| \le \epsilon$ whenever $n, m \ge N$.

5. Consider the problem of computing the root α: $f(\alpha) = 0$.

   (a) Derive the secant method

   $$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$

   (b) Show that

   $$\alpha - x_{n+1} = -(\alpha - x_n)(\alpha - x_{n-1})f[x_{n-1}, x_n, \alpha]/f[x_{n-1}, x_n].$$

6. The goal of this problem is the analysis of the stability of the following scheme (mid-point method)

   $$y_{n+1} = y_{n-1} + 2h\lambda y_n.$$

   (a) Show that this equation admits the roots

   $$r_0 = h\lambda + \sqrt{h^2\lambda^2 + 1}, \quad r_1 = h\lambda - \sqrt{h^2\lambda^2 + 1}.$$

   (b) Is the strong root condition satisfied?

   (c) Knowing that the general solution is given by $y_n = \beta_0 r_0^n + \beta_1 r_1^n$ where

   $$\beta_0 = \frac{y_1 - r_1 y_0}{r_0 - r_1}, \quad \beta_1 = \frac{y_0 r_0 - y_1}{r_0 - r_1}$$

   show using Taylor's theorem that

   $$\begin{cases} \beta_0 = \dfrac{e^{\lambda h} - r_1}{2\sqrt{1 + h^2\lambda^2}} = 1 + O(h^2\lambda^2) \\ \beta_1 = \dfrac{r_0 - e^{\lambda h}}{2\sqrt{1 + h^2\lambda^2}} = O(h^3\lambda^3). \end{cases}$$

   (d) Show that $y_n \to e^{\lambda x_n}$ when $h \to 0$.