Doctoral Qualifying Exam: Real and Complex Analysis

23 August, 2010

Problem 1. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined as follows:

$f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/3]}$, $f_5 = \chi_{[1/3,2/3]}$, $f_6 = \chi_{[2/3,1]}$, $f_7 = \chi_{[0,1/4]}$, $f_8 = \chi_{[1/4,1/2]}$, $f_9 = \chi_{[1/2,3/4]}$, $f_{10} = \chi_{[3/4,1]}$, $f_{11} = \chi_{[0,1/5]}$, $f_{12} = \chi_{[1/5,2/5]}, \ldots$ and so on,

where χ is the usual characteristic function on subsets.

(a) Explain why each function in the sequence is (Lebesgue) measurable.

(b) Determine whether or not the sequence $\{f_n\}$: (i) converges in measure; (ii) converges almost everywhere; (iii) converges; (iv) converges in the L^1 mean; (v) converges almost uniformly; (vi) converges uniformly, explaining your answer in each case.

Problem 2. Let $\{g_n\}$ be a sequence of (Lebesgue) measurable functions all having domains equal to a subset X of $\mathbb{R}^m$, where X is of finite (Lebesgue) measure. Suppose that $g_n \to g$ almost everywhere on X.

(a) Prove that for every $\epsilon > 0$ there exists a measurable subset Y of X and a measurable function g on X such that $m\,(X \setminus Y) < \epsilon$ and

$$\int_Y |g - g_n| \to 0 \text{ as } n \to \infty.$$

(b) Show that if no other assumptions are made, the result in (a) cannot be extended to all of X.

(c) Prove that there is a single additional assumption on the limit function g that allows extension of the result in (a) to all of X. Explain your answer.

Problem 3. Consider the integral equation

$$\varphi(t) = \int_0^\pi \sin\left(\lambda t \varphi(\tau)\right) d\tau, \tag{*}$$

(a) Prove that (*) has a unique continuous solution $\varphi : [0,\pi] \to \mathbb{R}$ for $0 < \lambda$ sufficiently small.

(b) Find the unique solution for each λ found in (a).

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(c) Show that (*) actually has a continuous solution for all $\lambda > 0$, but it is not unique for some values of λ such as $\lambda = 1$. (Hint: for non-uniqueness, look for the simplest possible solutions.)

Problem 4. By means of complex contour integration and the Residue theorem, show that

$$\int_0^\infty \frac{\log x}{1+x^4}\, dx = -\frac{\pi^2}{8\sqrt{2}}.$$

Identify the other real integral that the complex contour integral allows you to evaluate.

Problem 5. Let D be the square region

$$D = \{z : |\Re(z)| < 1,\ |\Im(z)| < 1\}$$

and suppose $f(z)$ is a function analytic on D and such that $f(z) = 0$ on one side of D: $f(z) = 0$ on $\Re(z) = 1$, $-1 < \Im(z) < 1$. By considering the function g defined by

$$g(z) = f(z)f(iz)f(-z)f(-iz),$$

or otherwise, prove that f is identically zero in D.
Hint: Show first that g must vanish on the boundary of the entire square, and then recall the maximum modulus theorem for analytic functions.

Problem 6. By considering the function $z^{n-1}e^{1/z-z}$ integrated around the unit circle (or otherwise), show that for $n = 1, 2, \ldots$,

$$\int_0^{2\pi} \cos(n\theta - 2\sin\theta)\, d\theta = 2\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!}.$$

Hint: Evaluate the integral two different ways.