Doctoral Qualifying Exam: Linear Algebra, Probability Distributions, and Statistical Inference

23 August, 2010

Problem 1.

(a) Find all 2×2 matrices A such that

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) Find all 2×2 matrices A such that

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 2. Suppose

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(a) Find the four fundamental subspaces (column space, null space, row space and left null space) of A.

(b) Find the set of all 3×3 real matrices that have the same fundamental subspaces as A.

Problem 3.

(a) Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

What are the eigenvalues of A?

(b) Let B be an $n \times n$ real symmetric matrix with all zeros on the diagonal ($B_{\ell\ell}$ for $\ell = 1, \ldots, n$). Furthermore, suppose that $I + B$ is positive definite. Prove that the largest eigenvalue of B is less than $n - 1$.

Problem 4. Let X and Y be random variables such that $E(X^k)$ and $E(Y^k) \neq 0$ exist for $k = 1, 2, 3, \ldots$. If the ratio X/Y and its denominator Y are independent, prove that $E[(X/Y)^k] = E(X^k)/E(Y^k), k = 1, 2, 3, \ldots$.

Problem 5. Let $X_1, \ldots, X_n$ be a random sample from

$$f(x; \theta) = \frac{exp\{-(x-\theta)\}}{(1 + exp\{-(x-\theta)\})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

Show that the likelihood equation has a unique solution $\hat{\theta}$ and the solution is a maximum. Assume that the regularity conditions hold. What can one say about the asymptotic properties of the estimator $\hat{\theta}$? Derive an $(1-\alpha)$ large sample confidence interval for θ_0 the true parameter.

Problem 6. Let $X_1, \ldots, X_n$ denote a random sample from a gamma distribution with $\alpha = 3$ and $\beta = \theta$. Let $H_0 : \theta = 2$ and $H_1 : \theta > 2$. The gamma density with parameters (α, β), $0 < \alpha, \beta < \infty$, is given by

$$f(x) = \begin{cases} \frac{x^{\alpha-1} exp\{-(x/\beta)\}}{\Gamma(\alpha)\beta^\alpha}, & 0 < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that there exists a uniformly most powerful test for H_0 against H_1, determine the statistics Y upon which the test may be based, and indicate the nature of the best critical region.

(b) Find the distribution of Y in Part(a) of this problem. If we want a significance level of 0.05, write an equation which can be used to determine the critical region. Express the power function as an integral.