Doctoral Qualifying Exam: Linear Algebra and Numerical Methods

23 August, 2010

Problem 1.

(a) Find all 2×2 matrices A such that

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) Find all 2×2 matrices A such that

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 2. Suppose

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(a) Find the four fundamental subspaces (column space, null space, row space and left null space) of A.

(b) Find the set of all 3×3 real matrices that have the same fundamental subspaces as A.

Problem 3.

(a) Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

What are the eigenvalues of A?

(b) Let B be an $n \times n$ real symmetric matrix with all zeros on the diagonal ($B_{\ell\ell}$ for $\ell = 1, \ldots, n$). Furthermore, suppose that $I + B$ is positive definite. Prove that the largest eigenvalue of B is less than $n - 1$.

Problem 4.

(a) (5 points) State the Chebyshev equioscillation theorem.

(b) (15 points) Let $f \in C^2[a,b]$ with $f''(x) > 0$ for $a \le x \le b$. Find the linear minimax approximation $q_1^*(x) = a_0 + a_1 x$ to $f(x)$ on $[a,b]$. Write down the explicit expressions for a_0 and a_1. What is the minimax error?

Problem 5.

(a) (5 points) Write down the definition of the spline function of order m.

(b) (15 points) Let $s(x)$ be a spline function of order m. Let b be a knot, and let $s(x)$ be a polynomial of degree $\le m-1$ on $[a,b]$ and $[b,c]$. Show that if $s^{(m-1)}(x)$ is continuous at $x = b$, then $s(x)$ is a polynomial of degree $\le m-1$ for $a \le x \le c$.

Problem 6.

(a) (10 points) Find all explicit fourth-order formulas of the form

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + h\left[b_0 y_n' + b_1 y_{n-1}' + b_2 y_{n-2}'\right].$$

(b) (10 points) Show that every such method is unstable.