Doctoral Qualifying Exam: Real Analysis and Probability

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1. Prove or disprove that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f([0,1]) = (0,1)$.

2. Let $\{a_n\}_{n=1}^{\infty}$ be sequence of numbers such that $a_n \in [0,1]$, $a_{n+1} < a_n$ and $a_n \to 0$ as $n \to \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, k times differentiable function such that $f(a_n) = 0$ for each n. Prove or disprove that $f^{(m)}(0) = 0$ for each $m \le k$.

3. Prove that the set Q of all rational numbers contained in the interval $[-10, 7]$ is Lebesgue measurable. Find its measure.

4. (i) If P is a probability measure on $(\Omega, \mathcal{A})$ and $\{A_n \in \mathcal{A};\ n = 1, 2, \cdots\}$ are events such that $A_n \downarrow A$, then show that,

   a) A is an event (i.e., $A \in \mathcal{A}$), and
   b) $P(A_n) \downarrow P(A)$.

   (ii) X is an integrable r.v. defined on a probability space $(\Omega, \mathcal{F}, P)$, and $\mathcal{A}$ is an "*almost trivial*" sub-σ-field contained in $\mathcal{F}$. Prove that $E(X|\mathcal{A})$ is *almost surely* a constant c, and that we must necessarily have $c = EX$.

5. (i) If $\{A_n;\ n = 1, 2, \cdots\}$, B and C are events defined on a probability space $(\Omega, \mathcal{F}, P)$, such that

   (C1) B and C differ by P-null sets, and
   (C2) there exists N such that $n \ge N \Rightarrow B \subset A_n \subset C$;
   then prove that,
   a) $\lim_{n\to\infty} P(A_n) = P(B) = P(C)$; and
   b) $P(\liminf_{n\to\infty} A_n) = P(\limsup_{n\to\infty} A_n)$.

   While $P(A_n)$ thus converges under conditions (C1) and (C2) above; does $\lim_{n\to\infty} A_n$ necessarily exist?

   (ii) Give a counterexample to show that convergence in probability does *not* imply convergence almost surely.

6. (i) Let $\{X_n\ ; n = 1, 2, \cdots\}$ be a sequence of r.v.s defined on a probability space, such that

$$X_n := \begin{cases} n^2, & \text{w.p. } \frac{1}{n^2} \\ -1, & \text{w.p. } \left(1 - \frac{1}{n^2}\right) \end{cases}$$

   Show that, as $n \to \infty$, $S_n := \sum_{i=1}^{n} X_i \longrightarrow -\infty$ a.s. Does X_n converge in distribution? If yes, what does it converge to?

   (ii) If $\{X_n\ ; n = 1, 2, \cdots\}$ and $\{Y_n\ ; n = 1, 2, \cdots\}$ are two sequences of r.v.s such that $|X_n| \le |Y_n|$ a.s. , for all $n \ge 1$; then :

a) Show that, $Y_n \xrightarrow{P} 0 \Longrightarrow X_n \xrightarrow{P} 0$.

b) If in fact, $Y_n \xrightarrow{\text{a.s.}} 0$; does it follow that $X_n \xrightarrow{\text{a.s.}} 0$ as well? Justify your answer.