Doctoral Qualifying Exam: Real and Complex Analysis

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1. Prove or disprove that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f([0,1]) = (0,1)$.

2. Let $\{a_n\}_{n=1}^{\infty}$ be sequence of numbers such that $a_n \in [0,1]$, $a_{n+1} < a_n$ and $a_n \to 0$ as $n \to \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, k times differentiable function such that $f(a_n) = 0$ for each n. Prove or disprove that $f^{(m)}(0) = 0$ for each $m \leq k$.

3. Prove that the set Q of all rational numbers contained in the interval $[-10, 7]$ is Lebesgue measurable. Find its measure.

4. *Complex integration.*

   (a) By evaluating a suitable contour integral, compute the value of

   $$\int_0^{2\pi} \frac{d\theta}{A + B\sin\theta}, \quad A^2 > B^2,\ A > 0.$$

   (b) Evaluate

   $$\int_0^{\infty} \frac{dx}{1 + x^{10}}$$

   (you may find it helpful to consider integrating around a sector).

5. *Liouville's theorem (may be assumed).*

   Let f be analytic on $\mathbb{C}$.

   (a) Prove that if $|f(z)| > M > 0$ in $\mathbb{C}$ then f is constant.

   (b) Prove (carefully) that if e^f is bounded then f is constant.

   (c) Prove that if $\mathrm{Re}(f)$ is bounded, either above or below, then f is constant. (Hint: consider how you might use (b).)

6. *Identity theorem.*

   The Identity Theorem states that if a function f is analytic on a region D, and if the set of zeros of f on D has a limit point in D, then the function f must be identically zero on D.

   Suppose that (c_n) is a sequence of complex numbers such that $\sum_n |c_n|$ converges, and

   $$\sum_{n=0}^{\infty} c_n k^{-n} = 0, \quad k = 1, 2, 3, \ldots.$$

   By defining a suitable analytic function and applying the identity theorem, prove that $c_n = 0\ \forall n$. You should justify all steps of your argument.