Doctoral Qualifying Exam: Linear Algebra, Probability Distributions and Statistical Inference

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Problem 1. Consider the vector space $\mathbf{R}^{n\times n}$ consisting of all $n \times n$ matrices with real entries. Define the function $L : \mathbf{R}^{n\times n} \to \mathbf{R}^{n\times n}$ defined by

$$L(X) \;=\; X^T.$$

(a) Show that L is an invertible linear mapping on $\mathbf{R}^{n\times n}$.
(b) Find the eigenvalues and corresponding eigenspaces of L.
(c) Find or explain why it is not possible to find a matrix $A \in \mathbf{R}^{n\times n}$ such that

$$L(X) \;=\; AX \qquad \text{for every } X \in \mathbf{R}^{n\times n}.$$

Problem 2. Let α be a real number. Consider the sequences of vectors $\mathbf{x}_\ell \in \mathbf{R}^n$ that satisfies the recursion

$$\mathbf{x}_{\ell+1} \;=\; 2\alpha\mathbf{x}_\ell \;-\; \alpha^2\mathbf{x}_{\ell-1}.$$

(a) Find $\mathbf{x}_\ell$ explicitly in terms of $\mathbf{x}_0$ and $\mathbf{x}_1$.
(b) Find the values of α for which the sequence $\mathbf{x}_\ell$ is bounded regardless of the values of $\mathbf{x}_0$ and $\mathbf{x}_1$.
(c) For $\alpha = 1$ find necessary and sufficient conditions on $\mathbf{x}_0$ and $\mathbf{x}_1$ for the sequence $\mathbf{x}_n$ to be bounded.

Problem 3. Let

$$A \;=\; \begin{pmatrix} 0 & 3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let r be the dimension of the row space and column space of A, Find an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ of the column space, an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of the row space and a set of nonnegative numbers $\{\sigma_1, \ldots, \sigma_r\}$ such that

$$A \;=\; \sum_{\ell=1}^{r} \sigma_\ell \mathbf{u}_\ell \mathbf{v}_\ell^T.$$

Problem 4. Let $X \sim \text{Uniform}(0, 1)$, and $0 < a < b < 1$. Set,

$$Y := \begin{cases} 1, & \text{if } 0 < X < b \\ 0, & \text{otherwise} \end{cases}$$

and

$$Z := \begin{cases} 1, & \text{if } a < X < 1 \\ 0, & \text{otherwise} \end{cases}$$

a) Are Y and Z statistically independent? Why/Why not?

b) Show that $E(Y|Z) = 1 - \left(\frac{1-b}{1-a}\right) Z$.

Problem 5.

(i) Consider a random variable (r.v.) X which cannot be observed unless it exceeds a value c. The *observed* value of X, is then fefined as a r.v. Y such that its distribution is the same as the conditional distribution of X given $\{X > c\}$; i.e.,

$$Y \stackrel{d}{=} X|X > c$$

Consider the case, where $c = 0$ and the r.v. X above, is *non-negative integer valued*, with a finite variance σ^2, mean μ, and $p_0 \equiv P(X = 0)$. Further suppose the moment generating function (mgf) $M_X(t)$ of X exists in a neighborhood of zero.

a) Find EY and $\text{var}(Y)$ in terms of μ, σ^2, and p_0.

b) Compute the m.g.f. of Y, and use it to verify your results in part a) above.

(ii) Construct the exact likelihood ratio test for testing $H_0 : \theta \leq \theta_0$ versus the alternatives $H_1 : \theta > \theta_0$, using a random sample of size n from the *shifted exponential* model with probability density

$$f(x|\theta) := \begin{cases} e^{-(x-\theta)}, & \text{if } x \geq \theta \\ 0, & \text{if } x < \theta \end{cases}$$

Write down the exact size-α critical region, for rejecting the null hypothesis.

Problem 6. Using a random sample $(X_1, X_2, \cdots, X_n)$ from a "binomial(k, θ)" population,with $n \geq 2$. Compute the unique *uniformly minimum variance unbised* (UMVU) estimator of

$$g(\theta) := P_\theta(X_1 = k) = \theta^k.$$