Doctoral Qualifying Exam: Linear Algebra and Numerical Methods

August 25, 2009

Problem 1. Consider the vector space $\mathbf{R}^{n\times n}$ consisting of all $n \times n$ matrices with real entries. Define the function $L : \mathbf{R}^{n\times n} \to \mathbf{R}^{n\times n}$ defined by

$$L(X) \;=\; X^T.$$

(a) Show that L is an invertible linear mapping on $\mathbf{R}^{n\times n}$.
(b) Find the eigenvalues and corresponding eigenspaces of L.
(c) Find or explain why it is not possible to find a matrix $A \in \mathbf{R}^{n\times n}$ such that

$$L(X) \;=\; AX \qquad \text{for every } X \in \mathbf{R}^{n\times n}.$$

Problem 2. Let α be a real number. Consider the sequences of vectors $\mathbf{x}_\ell \in \mathbf{R}^n$ that satisfies the recursion

$$\mathbf{x}_{\ell+1} \;=\; 2\alpha\mathbf{x}_\ell \;-\; \alpha^2\mathbf{x}_{\ell-1}.$$

(a) Find $\mathbf{x}_\ell$ explicitly in terms of $\mathbf{x}_0$ and $\mathbf{x}_1$.
(b) Find the values of α for which the sequence $\mathbf{x}_\ell$ is bounded regardless of the values of $\mathbf{x}_0$ and $\mathbf{x}_1$.
(c) For $\alpha = 1$ find necessary and sufficient conditions on $\mathbf{x}_0$ and $\mathbf{x}_1$ for the sequence $\mathbf{x}_n$ to be bounded.

Problem 3. Let

$$A \;=\; \begin{pmatrix} 0 & 3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let r be the dimension of the row space and column space of A, Find an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ of the column space, an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of the row space and a set of nonnegative numbers $\{\sigma_1, \ldots, \sigma_r\}$ such that

$$A \;=\; \sum_{\ell=1}^{r} \sigma_\ell \mathbf{u}_\ell \mathbf{v}_\ell^T.$$

Problem 4. Consider Newton's method for finding the positive square root of $a > 0$. Derive the following results, assuming $x_0 > 0$, $x_0 \neq \sqrt{a}$.

(a) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$

(b) $x_{n+1}^2 - a = \left[\dfrac{x_n^2 - a}{2x_n}\right]^2$, $n \geq 0$, and thus $x_n > \sqrt{a}$ for all n.
(c) The iterates $\{x_n\}$ are a strictly decreasing sequence for $n \geq 1$. Hint: Consider the sign of $x_{n+1} - x_n$
(d) $e_{n+1} = -\dfrac{e_n^2}{2x_n}$, with $e_n = \sqrt{a} - x_n$.
(e) $\mathrm{Rel}(x_{n+1}) = -\dfrac{\sqrt{a}}{2x_n}[\mathrm{Rel}(x_n)]^2$, with $\mathrm{Rel}(x_n)$ is the relative error in x_n.

Problem 5. Let $p_2(x)$ be the quadratic polynomial interpolating $f(x)$ at the evenly spaced points x_0, $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$.
(a) Derive formulas for the error $f'(x_i) - p_2'(x_i)$, $i = 0, 1, 2$.
(b) Bound these errors with $f(x) = \ln(1 + x)$. Hint: use the formula $f(t) - p(t) = (t - x_0) \ldots (t - x_n) f[x_0, \ldots, x_n, t]$.

Problem 6. Consider the explicit two-step second order method

$$y_{n+1} = 5y_n - 6y_{n-1} + \frac{h}{2}(f(x_n, y_n) - 3f(x_{n-1}, y_{n-1})) \quad x_0 \leq x_n \leq b \tag{1}$$

Consider the problem

$$y' = 0, \quad \text{with } y(0) = 0, \tag{2}$$

which has the solution $Y(x) = 0$. Applying (1) to (2) with $y_0 = y_1 = 0$ give the numerical solution $y_n = 0$, $n \geq 0$. Consider now the perturbed problem to (2) with initial data $z_0 = \varepsilon/2$ and $z_1 = \varepsilon$, and $\varepsilon \neq 0$:

a) Show that $z_n = \varepsilon\ 2^{n-1}$.

b) Show that
$$\max_{x_n} |y_n - z_n| = \max_{x_n} |\varepsilon| 2^{n-1} = |\varepsilon| 2^{N(h)-1}$$
where $N(h)$ is the largest subscript N for which $x_N \leq b$.

c) Show that (1) is unstable.

d) Recall the root condition and show that it is not satisfied.