Doctoral Qualifying Exam : Real and Complex Analysis
Monday August 25, 2008

1. Let $I = [a, b]$ be a compact interval. Suppose the sequence $f_n \to f$ uniformly on I and that each f_n is Riemann integrable on I. Show

(a) f is Riemann integrable on I.

(b) $lim_{n\to\infty} \int_a^x f_n(t)\, dt = \int_a^x f(t)\, dt$.

(c) Does the result in part b) hold if I is an unbounded interval? Explain or give a counterexample.

2. If $f \in M^+(X, \mathbf{X})$, $\lambda : \mathbf{X} \to \mathbf{R}$ such that $\lambda(E) = \int_E f\, d\mu$, prove λ is a measure.

3. Let $f(x)$ be a 2π-periodic function defined on $[-\pi, \pi]$ by $f(x) = \pi - |x|$, if $x \neq 0$ and $f(x) = 0$, if $x = 0$.

(a) Find the Fourier series generated by f.

(b) For which values of x, if any, does the Fourier series converge and to what value? Justify your results.

4. Find the Taylor or Laurent expansion about $z = 0$ for the function

$$f(z) = \frac{1}{(z^2 + 1)(z + 2)},$$

when (i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$.

5. Evaluate

$$\int_0^\infty \frac{t^{a-1}}{1 + t}\, dt,$$

where $0 < a < 1$ by the method of contour integration.

6. Let G be a bounded region and suppose f is analytic in G and continuous on the closure of G.

(a) Show that if c is a nonnegative constant and $|f| = c$ in G, then f is constant in G (Hint: use the Cauchy-Riemann equations).

(b) Use the result in (a) to show that if $|f| = c$ for all z on the boundary of G then either f is a constant or f has a zero in G.