Doctoral Qualifying Exam : Linear Algebra, Probability Distributions and Statistical Inference
Wednesday August 27, 2008

1. Let
$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

(a) Find the LU-factorization of A.

(b) Diagonalize A.

(c) Find the singular value decomposition of A.

2. Let A be an $n \times n$ real symmetric matrix.

(a) Prove that all the eigenvalues of A are real.

(b) Prove that each diagonal element of A lies in the closed interval $[\lambda_1, \lambda_n]$ where λ_1 is the smallest eigenvalue of A and λ_n is the largest eigenvalue of A.

3. Let U and V be subspaces of $\mathbf{R}^n$. Let P_U and P_V be the $n \times n$ matrices that represent orthogonal projection into U and V respectively. Let $W = U \cap V$.

(a) Prove W is a subspace of $\mathbf{R}^n$.

(b) Prove that
$$S = \lim_{n\to\infty} (P_U P_V)^n$$
exists and that S represents orthogonal projection into W.

4. This question has two independent parts.

(a) Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(\theta, \theta + 1), -\infty < \theta < \infty$. Obtain the probability density function of the range statistic and hence show that the range is an ancillary statistic.

(b) Suppose that $X_1, \ldots, X_n$ denote a random sample from a uniform distribution on $(0, 1)$. Let $Y_n = \max_{1\le i\le n} X_i$. Obtain the limit distribution of $n(Y_n - 1)$.

5. This question has two independent parts.

(a) For any two random variables X an Y with finite variances, show that

(i) $\text{Cov}(X, Y) = \text{Cov}(X, E(X|Y))$.

(ii) X and $Y - E(Y|X)$ are uncorrelated.

(iii) $\text{Var}(Y - E(Y|X)) = E(\text{Var}(Y|X))$.

(b) Let X_p denote a random variable with a chi squared distribution having p degrees of freedom. For any function $h(x)$, show that $E(h(X_p)) = pE(X_{p+2})/X_{p+2})$, provided the expectations exist.

6. Let $f(x|\theta)$ denote the logistic location family given by

$$f(x|\theta) = \frac{e^{x-\theta}}{(1 + e^{x-\theta})^2}, -\infty < x < \infty, -\infty < \theta < \infty.$$

(a) Based on one observation X, obtain the most powerful size α test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. For $\alpha = 0.2$, find the size of the Type II Error.

(b) Show that the test in part (a) is UMP size α test for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$.