Doctoral Qualifying Exam : Linear Algebra and Numerical Methods
Wednesday August 27, 2008

1. Let

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

(a) Find the LU-factorization of A.

(b) Diagonalize A.

(c) Find the singular value decomposition of A.

2. Let A be an $n \times n$ real symmetric matrix.

(a) Prove that all the eigenvalues of A are real.

(b) Prove that each diagonal element of A lies in the closed interval $[\lambda_1, \lambda_n]$ where λ_1 is the smallest eigenvalue of A and λ_n is the largest eigenvalue of A.

3. Let U and V be subspaces of $\mathbf{R}^n$. Let P_U and P_V be the $n \times n$ matrices that represent orthogonal projection into U and V respectively. Let $W = U \cap V$.

(a) Prove W is a subspace of $\mathbf{R}^n$.

(b) Prove that

$$S = \lim_{n \to \infty} (P_U P_V)^n$$

exists and that S represents orthogonal projection into W.

4.

(a) Show there is a unique cubic polynomial $p(x)$ for which

$$p(x_0) = f(x_0), \qquad p(x_2) = f(x_2)$$
$$p'(x_1) = f'(x_1), \qquad p''(x_1) = f''(x_1)$$

where $f(x)$ is a given function and $x_0 \neq x_2$.

(b) Let $f(x) = \cos^{-1}(x)$ for $-1 \leq x \leq 1$ (the principal branch $0 \leq f \leq \pi$). Find the polynomial of degree at most two,

$$p(x) = a_0 + a_1 x + a_2 x^2$$

which minimizes

$$\int_{-1}^{1} \frac{[f(x) - p(x)]^2}{\sqrt{1 - x^2}} dx.$$

5.

(a) Determine the degree of precision of the following quadrature formula:

$$\int_0^\infty f(t)e^{-t}dt = \frac{5}{3}f(1) - \frac{3}{2}f(2) + f(3) - \frac{1}{6}f(4).$$

(b) Derive the two-point Gaussian quadrature formula for

$$I = \int_0^1 xf(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

with weight function $w(x) = x$. Find w_1, w_2, x_1, x_2.

6. The general Runge-Kutta method for solving the initial value problem $y' = f(t, y)$ takes the following form:

$$\xi_i = y_n + h\sum_{i=1}^{\nu} a_{j,i} f(t_n + c_i h, \xi_i), \qquad j = 1, 2, \cdots, \nu$$

$$y_{n+1} = y_n + h\sum_{j=1}^{\nu} b_j f(t_n + c_j h, \xi_j),$$

where $a_{j,i}$, b_j, c_i $(j, i = 1, 2, \cdots, \nu)$ are the RK matrix, weights, and nodes respectively.

(a) When is the Runge-Kutta method explicit?

(b) Suppose that the Runge-Kutta method is used to solve the model problem $y' = \lambda y$, $y(0) = 1$. Show that the numerical solution has the following form

$$y_n = [r(h\lambda)]^n, \qquad n = 0, 1, \cdots$$

where $r(h\lambda)$ is a rational function of $h\lambda$ in general. Write down an explicit expression of $r(h\lambda)$ using RK matrix and weights.

(c) Show that if the Runge-Kutta method is explicit then r is a polynomial of $h\lambda$.