Doctoral Qualifying Exam: Linear Algebra, Probability Distributions and Statistical Inference
Monday, August 29, 2007

Problem 1

Given the 3×2 matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \\ 2 & 2 \end{pmatrix}.$$

Find bases for the four fundamental subspaces. Find the singular value decomposition (SVD) of this matrix and decompose the matrix into the form $\mathbf{A} = \sum_{i=1}^{2} \alpha_i \mathbf{u}_i \mathbf{v}_i^T$, where $\mathbf{u}_1^T = (1, 0, 1)$, $\mathbf{u}_2^T = (0, 1, 0)$,$\mathbf{v}_1^T = (1, 1)$, and $\mathbf{v}_2^T = (1, -1)$.

Problem 2

Find the Jordan canonical form $\mathbf{J}$ for the following matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -2 \\ -1 & -2 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Solve the differential equation $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, with $\mathbf{x}(0) = (1 \quad 1 \quad 1)^T$.

Problem 3

Let $\mathbf{A}$ be an $n \times n$ matrix. Let $\rho_1, \ldots, \rho_n$ be any n positive numbers. Show that every eigenvalue, λ, of $\mathbf{A}$ must satisfy at least one of the inequalities

$$|\lambda - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} (\rho_i / \rho_j)|a_{ij}| \qquad (i = 1, \ldots, n).$$

Problem 4

Consider a random sample $X_1, X_2, \ldots, X_n$ from a normal distribution with mean μ and variance σ^2. Both μ and σ^2 are unknown. Find the exact likelihood ratio test of hypothesis $H_0 : \mu = \mu_0$ versus $H_a : \mu \neq \mu_0$. Show that this test is given by $|t| \geq c$, where $t = \frac{\bar{X} - \mu_0}{\frac{S}{\sqrt{n}}}$ and $S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n - 1}$.

Problem 5

Let $Y_1, Y_2, \ldots, Y_n$ denote ar random sample from the exponential denisty fuction given by $f(y, \theta) = \left(\frac{1}{\theta}\right) \exp^{\frac{-y}{\theta}} I_{(0,\infty)}(y)$, where $\theta > 0$. Find the uniformly minimum variance unbiased estimator of $Var(Y_1)$.

Problem 6

Let X and Y be two independent uniformly distributed discrete random variables on $\{1, 2, ..., m\}$. Show that $E|X - Y| = \frac{(m-1)(m+1)}{3m}$.