Doctoral Qualifying Exam: Linear Algebra and Numerical Methods
Monday, August 29, 2007

Problem 1

Given the 3×2 matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \\ 2 & 2 \end{pmatrix}.$$

Find bases for the four fundamental subspaces. Find the singular value decomposition (SVD) of this matrix and decompose the matrix into the form $\mathbf{A} = \sum_{i=1}^{2} \alpha_i \mathbf{u}_i \mathbf{v}_i^T$, where $\mathbf{u}_1^T = (1,0,1)$, $\mathbf{u}_2^T = (0,1,0)$,$\mathbf{v}_1^T = (1,1)$, and $\mathbf{v}_2^T = (1,-1)$.

Problem 2

Find the Jordan canonical form $\mathbf{J}$ for the following matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -2 \\ -1 & -2 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Solve the differential equation $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, with $\mathbf{x}(0) = (1 \quad 1 \quad 1)^T$.

Problem 3

Let $\mathbf{A}$ be an $n \times n$ matrix. Let $\rho_1, \ldots, \rho_n$ be any n positive numbers. Show that every eigenvalue, λ, of $\mathbf{A}$ must satisfy at least one of the inequalities

$$|\lambda - a_{ii}| \le \sum_{j=1, j\neq i}^{n} (\rho_i/\rho_j)|a_{ij}| \qquad (i = 1, \ldots, n).$$

Problem 4

Let $\Psi(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$, and $\rho_n = \min ||\Psi(x)||_\infty$ where the minimization is over all interpolation nodes $\{x_j\}$, $j = 0, 1, \ldots, n$ and the infinity norm is taken on $-1 \le x \le 1$.

(a) What is ρ_n?

(b) What are the interpolation nodes $\{x_j\}$, $j = 0, 1, \ldots, n$ to achieve this minimal value ρ_n?

(c) Now let $\{x_j\}$, $j = 0, 1, \ldots, n$ denote the equispaced nodes on $[-1, 1]$, i.e., $x_j = -1 + \frac{2j}{n}$. Show that for this choice of nodes and for large n

$$\frac{||\Psi(x)||_\infty}{\rho_n} > \frac{\sqrt{2}}{n}\left(\frac{4}{e}\right)^n. \qquad (0.1)$$

(Hint: Consider $\Psi(-1+\frac{1}{n})$ and use Stirling's formula $n! \sim e^{-n}n^n\sqrt{2\pi n}$.)

Problem 5

Cauchy's pricinple value integral is defined via the formula

$$I(f) = \text{p.v.} \int_{-1}^{1} \frac{f(x)}{x} dx = \lim_{\epsilon \to 0+} \int_{-1}^{-\epsilon} \frac{f(x)}{x} dx + \int_{+\epsilon}^{1} \frac{f(x)}{x} dx. \tag{0.2}$$

(a) Suppose that we approximate $I(f)$ by

$$I_\epsilon(f) = \int_{-1}^{-\epsilon} \frac{f(x)}{x} dx + \int_{+\epsilon}^{1} \frac{f(x)}{x} dx. \tag{0.3}$$

Derive an error estimate for $I_\epsilon(f)$ when f is sufficiently smooth.
(b) Suppose that we can evaluate $I_\epsilon(f)$ very accurately. How do you obtain a better estimate for $I(f)$ using $I_\epsilon(f)$, $I_{\frac{\epsilon}{2}}(f)$, and $I_{\frac{\epsilon}{4}}(f)$? Derive an expression using $I_\epsilon(f)$, $I_{\frac{\epsilon}{2}}(f)$, and $I_{\frac{\epsilon}{4}}(f)$ for such estimate.

Problem 6

Consider the theta method for solving the initial value problem:

$$y_{n+1} = y_n + h[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})], \tag{0.4}$$

where θ is a fixed number in $[0, 1]$.
(a) Find the order of this scheme.
(b) Determine all values of θ such that the theta method is A-stable.