Doctoral Qualifying Examination in Analysis

Tuesday, August 30, 2005

Problem 1.

(a) Suppose that $K(x)$ is a continuously differentiable real-valued function satisfying $K(0) = 1$ and $1 < K'(x) < 2$ for $x \in \mathbf{R}$. Moreover, suppose that λ is a nonzero real constant. Show that if λ is sufficiently small then the nonlinear integral equation

$$\varphi(x) \;=\; \lambda \int_0^1 K\big(e^x \varphi(y)\big)\, dy$$

has a unique solution $\varphi(x)$.

(b) Show that the uniform limit of a sequence of continuous functions is continuous.

Problem 2. Suppose that $f \in L^1(\mathbf{R})$. Prove that

$$\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} f(x) e^{i\lambda x}\, dx \;=\; 0.$$

Hint: First prove the result for step functions.

Problem 3. Consider the function

$$f(x) \;=\; x(1-x)$$

on the interval $[-\pi, \pi]$.

(a) For $N \in \mathbf{N}$ find the function $f_N(x)$ in the span of $\{1, \cos x, \cos 2x, \cos 3x, \ldots, \cos Nx\}$ that minimizes

$$||f - f_N||^2 \;=\; \int_{-\pi}^{\pi} |f(x) - f_N(x)|^2\, dx.$$

(b) Find

$$\lim_{N \to \infty} f_N(x).$$

Problem 4. Sum the series

$$S = \sum_{n=1}^{\infty} \frac{n}{n^3 + n}$$

by writing the series as an integration in the complex plane and then using complex integral methods.

Problem 5. Evaluate the integral

$$I = \frac{1}{2\pi i} \int_{C_0} \frac{dz}{3z^3 + 2z^2 + z}$$

where C_0 is a square, centered at the origin, with sides parallel with the coordinate axes. The length of the side of the square is 4.

Problem 6. In the complex η-plane ($\eta = \eta' + i\eta''$), let C_0 be a smooth contour in the first quadrant which starts at $\eta = 1$ and ends at $\eta = i\infty$.

(a) Sketch such a curve.

(b) Choose a parametric representation for C_0 (let t be the parameter). $I(z)$ is defined by

$$I(z) = \frac{1}{2\pi i} \int_{C_0} \frac{e^{z\eta}}{\eta + (1 + i)}\, d\eta.$$

(c) For what values of the complex variable z does the integral converge.

(d) For what values of z is $I(z)$—defined by this integral—analytic.