Linear Algebra, Probability Distribution Theory, and Statistical Inference

Monday, August 29, 2005

You have three hours for this exam. Show all work in the answer books provided. The six questions carry equal weight.

1) A linear operator T on an n-dimensional vector space V is called an isometry if $||T(v)|| = ||v||$ for all $v \in V$. Show that T is an isometry on V if and only if T maps an orthonormal basis of V onto on orthonormal basis of V.

2) Let A be an $n \times n$ symmetric matrix. If λ_i and λ_j are distinct eigenvalues of A, with associated eigenvectors v_i and v_j, show that v_i is orthogonal to v_j.

3) (a) Use Gerschgorin's Theorem to prove that the following matrix has 3 real eigenvalues.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1/2 & 6 & 1/2 \\ 2 & 0 & 8 \end{bmatrix}$$

(b) Given the matrix A below, find a matrix B such that $J = B^{-1}AB$ is in Jordan Canonical Form.

$$A = \begin{bmatrix} 17 & -25 & 0 \\ 9 & -13 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

4) Let $X_1, \cdots, X_n$ be a random sample from the Normal distribution $N(\mu, \sigma^2)$, with unknown mean (μ) and unknown variance (σ^2).

(i) Find an unbiased estimator of the standard deviation σ.

(ii) Let ξ_p be the *upper* p-th quantile (i.e., the unique solution of the equation $P(X_1 \geq \xi_p) = p$), where $p \in (0,1)$ is fixed and known. Find the uniformly minimum variance unbiased (UMVU) estimator of ξ_p. Why is it unique ?

5) Suppose $X_1, \cdots, X_n$ is a random sample from the binomial distribution $b(1, \theta)$.

(i) Find the maximum likelihood estimator (MLE) of θ, if it is known apriori that $\theta \in \left(\frac{1}{4}, \frac{3}{4}\right)$.

(ii) Construct an exact likelihood ratio (LR) test of the null hypothesis $H_0 : \theta \in \left(\frac{1}{4}, \frac{3}{4}\right)$ vs. the composite alternative $H_1 : \theta \leq \frac{1}{4}$, or $\theta \geq \frac{3}{4}$.

Questions continued over page...

6) Suppose the random vector $(X_1, X_2)'$ has a bivariate Normal distribution $BN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, where X_1, X_2 have correlation ρ and, $EX_i = \mu_i,\ \ \text{var}X_i = \sigma_i^2; \quad i = 1, 2.$

(i) Prove that the random variables

$$U := \frac{\left(\frac{X_1-\mu_1}{\sigma_1}\right) - \rho\left(\frac{X_2-\mu_2}{\sigma_2}\right)}{\sqrt{(1-\rho^2)}} \quad \text{and} \quad V := \left(\frac{X_2-\mu_2}{\sigma_2}\right)$$

are independent and identically distributed as $N(0, 1)$.

(ii) Use the result in part (i) above to compute the joint moment generating function (m.g.f.) of (X_1, X_2). (You may assume the expression for the m.g.f. of a standard Normal distribution.)