

Doctoral Qualifying Exam: Applied Mathematics

Monday, August 29, 2005

You have three hours for this exam. Show all work in the answer books provided. The questions carry equal weight.

1) In a fishery, the population $N(t)$ satisfies the logistic equation

$$\frac{dN}{dt} = \alpha N \left(1 - \frac{N}{K}\right)$$

where α and K are positive constants. Assume that fish are caught at a constant rate r that is independent of the size of the fish population.

- What is the equation for dN/dt when $r > 0$?
- When fish are removed at a rate $r < \alpha K/4$, determine the equilibrium fish population(s).
- Determine the stability of these equilibrium population(s).
- When fish are caught at a rate $r > \alpha K/4$, show that $N(t)$ decreases to zero regardless of the initial size of the fish population.

2) Consider the boundary value problem

$$u'' + 2u' + u = f(x) \quad x \in (0, 1)$$

$$u'(0) + \alpha u(0) = c_1 \quad u(1) = c_2$$

where the parameter α is real.

State whether the problem is self-adjoint, formally self-adjoint, or neither, and explain why. Find the values of α for which a Green's function exists. Construct the Green's function when it exists, and give the solution of the boundary value problem in terms of it.

What condition, if any, must be satisfied by the data $(f(x), c_1, c_2)$ to ensure that a solution exists when $\alpha = 2$?

3) For the boundary value problem

$$(L - \mu)u \equiv \left(-\frac{d^2}{dx^2} - \mu\right)u = f(x) \quad x \in (0, 1)$$

$$u(0) = c_1 \quad u'(1) = c_2$$

where μ is a parameter, write down the associated problem for the eigensystem (i.e., the eigenfunctions and eigenvalues). Find the eigensystem explicitly, and use this to find the eigenfunction expansion of the solution to the boundary value problem. Comment on the convergence of your solution for $x \in [0, 1]$.

What happens, in general, to the solution as $\mu \rightarrow (n + \frac{1}{2})^2 \pi^2$ for some $n = 0, 1, \dots$, and under what circumstances may this *not* occur?

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4) Consider the partial differential equation

$$u_{xx} + (1 + y)^2 u_{yy} = 0.$$

(a) Classify the equation.

(b) Determine the characteristic equation for this partial differential equation.

(c) Rewrite the partial differential equation in canonical form by defining new independent coordinates.

5) (a) Use the method of images to find the Green's function in the quarter plane, $0 < x < \infty$, $0 < y < \infty$, satisfying

$$\begin{aligned} G_{\xi\xi} + G_{\eta\eta} &= \delta(\xi - x, \eta - y), & 0 < \xi < \infty, & 0 < \eta < \infty, \\ G(\xi, 0) &= 0, & 0 < \xi < \infty, & \\ \frac{\partial G}{\partial n}(0, \eta) &= 0, & 0 < \eta < \infty. & \end{aligned}$$

(b) Use the Green's function method to obtain a solution of the following problem in terms of integrals over the Green's function, G , obtained in part (a).

$$\begin{aligned} \nabla^2 u &= \phi(x, y), & 0 < x < \infty, & 0 < y < \infty, \\ u(x, 0) &= f(x), & 0 < x < \infty, & \\ \frac{\partial u}{\partial n}(0, y) &= g(y), & 0 < y < \infty & \end{aligned}$$

where ϕ , f , and g are given functions.

6) Solve the initial-boundary-value problem for the heat equation:

$$\begin{aligned} u_t &= k^2 u_{xx}, & 0 < x < \ell, & t > 0, \\ u(x, 0) &= 0, & 0 < x < \ell, & \\ u(0, t) &= 0, & u(\ell, t) = e^{-t}, & t > 0. \end{aligned}$$