Linear Algebra and Numerical Methods

Monday, August 29, 2005

You have three hours for this exam. Show all work in the answer books provided. The six questions carry equal weight.

1) A linear operator T on an n-dimensional vector space V is called an isometry if $||T(v)|| = ||v||$ for all $v \in V$. Show that T is an isometry on V if and only if T maps an orthonormal basis of V onto on orthonormal basis of V.

2) Let A be an $n \times n$ symmetric matrix. If λ_i and λ_j are distinct eigenvalues of A, with associated eigenvectors v_i and v_j, show that v_i is orthogonal to v_j.

3) (a) Use Gerschgorin's Theorem to prove that the following matrix has 3 real eigenvalues.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1/2 & 6 & 1/2 \\ 2 & 0 & 8 \end{bmatrix}$$

(b) Given the matrix A below, find a matrix B such that $J = B^{-1}AB$ is in Jordan Canonical Form.

$$A = \begin{bmatrix} 17 & -25 & 0 \\ 9 & -13 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

4) (a) Let $p_n(x)$ be the interpolating polynomial to a sufficiently smooth function $f(x)$ at $n+1$ distinct nodes $x_i \in (a,b)$, $i = 0, 1, ..., n$. Show that for each $x \in (a,b)$, there exists $\eta \in (a,b)$ such that

$$f(x) - p_n(x) \quad = \quad \frac{1}{(n+1)!} f^{(n+1)}(\eta) \prod_{i=0}^{n} (x - x_i).$$

(b) Show that for the Chebyshev nodes $x_i = \cos(\frac{(2i+1)\pi}{2n+2})$ and $x \in [-1,1]$, we have

$$\max_{-1 \le x \le 1} \prod_{i=0}^{n} (x - x_i) = 2^{-n}.$$

(c) Let $p_n(x)$ be the interpolating polynomial to $\ln(x+2)$ at the above Chebyshev nodes $x_{i\,i=0}^{\,n}$. Determine n such that

$$\max_{-1 \le x \le 1} |\ln(x+2) - p_n(x)| \le 2^{-24}.$$

Questions continued over page...

5) Consider the initial value problem

$$\begin{aligned} y' &= f(x, y), \\ y(x_0) &= y_0. \end{aligned}$$

The following method has been proposed as a means of numerically approximating the solution to this equation:

$$y_{n+2} - y_n = \frac{2}{3} h[f(t_{n+2}, y_{n+2}) + f(t_{n+1}, y_{n+1}) + f(t_n, y_n)],$$

where h is the step size.
(a) What is the order of this method?
(b) Is this method A-stable? Discuss the numerical stability of this method.

6) Consider the iteration method

$$x^{(k+1)} = Mx^{(k)} + b,$$

where $x^{(k)}$ and b are vectors in R^n, $M \in R^{n \times n}$ is a square matrix, and $x^{(0)}$ is a given initial guess. Assume that $||M|| < 1$, where $||M||$ is a matrix norm induced by the vector norm $||x||$. Show that:
(a) The process is convergent to the unique solution of the linear system $x = Mx + b$.
(b) Show that the error is bounded

$$||x^{(k)} - x|| \leq ||(I - M)^{-1}|| \cdot ||x^{(k+1)} - x^{(k)}||.$$

(c) Show also that, from the iteration relation and the result in (b),

$$||x^{(k)} - x|| \leq ||M||^k ||x^{(0)}|| + \frac{||M||^k \cdot ||b||}{1 - ||M||}.$$