Ph.D. Qualifying Exam in Analysis

August 25, 2004

Problem 1. Let $f_n(x) = x^n$. Let $g(x)$ be continuous on $[0,1]$ such that $g(1) = 0$.

(a) Prove that the sequence $\{f_n\}$ converges pointwise but not uniformly on $[0,1]$.

(b) Prove that the sequence $\{g(x)x^n\}$ converges uniformly on $[0,1]$.

Problem 2. Suppose $f_n \to f$ almost everywhere in some measure space $(X, \mathbf{X}, \mu)$.

(a) Does $f_n \to f$ in L_p? If so, prove it; if not, give a counterexample.

(b) If $\mu(X) < \infty$, does $f_n \to f$ in L_p? If so, prove it; if not, give a counterexample.

(c) If $|f_n| \leq g$ for all n and $g \in L_p$, does $f_n \to f$ in L_p? If so, prove it; if not, give a counterexample.

Problem 3. Consider the $2\pi-$periodic function defined by

$$f(x) = \begin{cases} x & x \in [0, \pi) \\ x - \pi & x \in [\pi, 2\pi) \end{cases}$$

(a) Compute the Fourier Series generated by f.

(b) Does it converge at any values of $[0, 2\pi]$? If so, to what value? Justify your results.

(c) Does it exhibit Gibbs phenomena? Explain.

Problem 4. (a) Locate and classify the singular points of the function

$$f(z) = \frac{1}{\sinh z}.$$

(b) Find the first three terms in the Laurent series expansion of $f(z)$ in the domain $0 < |z| < \pi$

(c) Calculate

$$\frac{1}{2\pi i} \int_c \frac{e^{zt}}{\sinh z} \, dz$$

where C is the circle $|z| = 8$ oriented counterclockwise.

Problem 5. Use contour integration to compute

$$\int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4}\, dx.$$

(Justify your result.)

Problem 6. Let $f(z)$ be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = az$, where a is a complex constant.