Linear Algebra & Numerical Methods Qualifying Exam - August 2004

1. (a) Show that the vectors $v^1 = col(1,1,0)$, $v^2 = col(2,0,1)$, and $v^3 = col(1,-1,2)$ are a basis for R^3.

   (b) Use the Gram-Schmidt procedure to construct an orthonormal basis $\{u^1, u^2, u^3\}$ from the basis given in (a), starting with $u^1 = v^1/||v^1||$ and using the standard inner product.

2. Let K and M be $n \times n$ Hermitian matrices, and let M be positive definite. Let the vectors $c^1, c^2, \ldots, c^n$ and the real numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ satisfy

$$Kc^j = \lambda_j M c^j, \qquad \left(Mc^j, c^k\right) = \delta_{jk}.$$

   Prove that

$$\lambda_1 = \max_{x \neq 0} \frac{(Kx, x)}{(Mx, x)},$$

   where x is any n-vector. Show that the maximum is achieved when $x = c^1$.

3. In the following, consider $n \times n$ matrices A.

   (a) Show that if A has a complete set of orthogonal eigenvectors, then $AA^* = A^*A$.

   (b) Use the fact that $AA^* = A^*A$ implies that A has a complete set of orthogonal eigenvectors to prove that Hermitian, skew-Hermitian, and unitary matrices have complete sets of orthogonal eigenvectors.

   (c) Do all matrices A with distinct eigenvalues satisfy $AA^* = A^*A$? Explain.

4. Consider the initial value problem

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0. \end{aligned}$$

   In order to numerically approximate the solution to this equation, the following method has been proposed:

$$y_{n+1} = -\frac{3}{2}y_n + 3y_{n-1} - \frac{1}{2}y_{n-2} + 3hf(x_n, y_n)$$

   where h is the step size.

   a. What is the order of this method?

   b. Discuss the numerical stability of this method.

5. Consider the following method for solving an $n \times n$ linear system. First, Gaussian elimination is used to reduce the linear system to triangular form. Then, the nth equation is used to eliminate x_n from each of the first $n-1$ equations, followed by the $(n-1)$th equation being used to eliminate x_{n-1} from each of the first $n-2$ equations, etc. This procedure is continued until the system is in diagonal form. The resulting diagonal system is then solved using division.

   Give an operation count for the use of the method described above to solve an $n \times n$ linear system. Assume that neither pivoting nor scaling is used. Please keep the additions/subtractions counted separately from the multiplications/divisions. Recall that

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

6. This problem involves the derivation of numerical derivative formulas through the use of interpolation.

   a. Find a polynomial $p(x)$ of degree ≤ 2 that interpolates a smooth function $f(x)$ with

$$\begin{aligned} p(x_0) &= f(x_0) = y_0, \\ p(x_1) &= f(x_1) = y_1, \\ p'(x_1) &= f'(x_1) = y_1', \end{aligned}$$

   where the nodes $\{x_0, x_1\}$ are distinct and the values $\{y_0, y_1, y_1'\}$ are given. Write $p(x)$ in the form

$$p(x) = y_0 g_0(x) + y_1 g_1(x) + y_1' g_2(x).$$

   b. What is the error in the using $p(x)$ to approximate $f(x)$?

   c. Assuming that $x_1 - x_0 = h$, derive the formula associated with approximating $f'(x_0)$ by $p'(x_0)$.

   d. Derive an error formula for the approximation in part **c**.