Doctoral Qualifying Exam: Real and Complex Analysis

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1. In this problem, all integrals are understood in the sense of Lebesgue.

   (a) Let $\Omega = [0, 1]$. Construct, if possible, a sequence of non-negative functions $\{f_n\}$ converging to a limit f a.e in Ω, such that

   $$\int_\Omega f < \liminf_{n\to\infty} \int_\Omega f_n.$$

   (b) State the Monotone Convergence Theorem.

   (c) Consider a series $\sum_{k=1}^{\infty} a_k(x)$, where $a_k(x) \geq 0$ are measurable for each k. Prove or disprove, that

   $$\int \sum_{k=1}^{\infty} a_k(x)dx = \sum_{k=1}^{\infty} \int a_k(x)dx.$$

2. Let $f \in L^\infty(\mathbb{R})$. Consider a mapping $f(x) \mapsto \mathcal{A}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy$.

   (a) Show that $\mathcal{A}f(x)$ maps $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$.

   (b) Give the definition of a contraction map.

   (c) Prove or disprove that $\mathcal{A}f(x)$ is a contraction in $L^\infty(\mathbb{R})$.

3. Let f be a Riemann integrable function on the unit circle with Fourier coefficients a_n, that is, $f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$.

   (a) Prove the best approximation theorem, that is, show that for any fixed integer N and any complex numbers c_n the following inequality holds:

   $$||f - \sum_{|n|\leq N} a_n e_n|| \leq ||f - \sum_{|n|\leq N} c_n||,$$

   where $||\cdot||$ denotes the L^2 norm and $e_n(x) = e^{inx}$. Moreover, equality holds if and only if $a_n = c_n$.

   (b) State Parseval's identity.

   (c) Prove or disprove that

   $$\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta \to 0 \qquad \text{as} \qquad n \to \infty.$$

4. In what regions of the complex plane do the following series converge absolutely (and thus define analytic functions?)

$$\text{(a)}\quad F(z) = \sum_{n=1}^{\infty} 2^{-n} \sin nz \qquad\qquad \text{(b)}\quad G(z) = \sum_{n=1}^{\infty} n^{-2} \sin nz.$$

It may help to use the exponential representation of the sine function when estimating the size of terms in the series.

5. Let the function f be analytic on the closed disk $D = \{z \in \mathbb{C} : |z| \leq R\}$, where $R > 0$. Suppose that f has only the zeros $z_1, \ldots, z_n$ in the the interior of D, with corresponding respective multiplicities $m_1, \ldots, m_n$, and does not vanish on the boundary of D. Prove that

$$\int_C \frac{z f'(z)}{f(z)} dz = 2\pi i \sum_{k=1}^{n} m_k z_k,$$

where C is the circle $|z| = R$ with counterclockwise orientation.

6. Using the Identity Theorem (or otherwise), show that there exists a unique analytic function $f(z)$, defined on a domain D containing the origin, such that

$$f\left(\frac{1}{n}\right) = (n^2 - 1)^{-1} \quad \text{for all but finitely many } n \in \mathbb{N}.$$

Give $f(z)$.