Doctoral Qualifying Exam : Real Analysis and Probability
Monday June 12, 2008

1.

(a) Let $X = \mathbf{R}$, $\mathbf{X} = \mathbf{B}$ and the Lebesgue measure. Consider the sequence of functions $f_n(x) = \chi_{[n,n+1/n]}$. Does this sequence converge uniformly, almost everywhere, in L_p, in measure or almost uniformly? In each case prove it or explain why not.

(b) Give an example of a function that converges almost everywhere, but not in measure.

(c) Prove that almost everywhere convergence implies convergence in meausure on a finite measure space.

2. Let X by a metric space and $\mathcal{C}(X)$ be the set of all continuous and bounded functions with domain X. If $f, g \in \mathcal{C}(X)$ define the distance between these functions by

$$d(f,g) = \sup_{x \in X} |f(x) - g(x)|.$$

(a) Prove that d is a metric on C.

(b) Prove that with this definition of distance $\mathcal{C}(X)$ is a complete metric space.

3.

(a) Express the Fourier series generated by x on $[0, 2\pi]$ in the form $\Sigma c_n \phi_n$ where ϕ_n are orthonormal on the set. Clearly identify the values of c_n and ϕ_n.

(b) From (a), calculate $\Sigma_{n=1}^{\infty} 1/n^2$ using Parseval's formula.

(c) Using the above, find a function defined on $[0, 2\pi]$ whose Fourier series equals $\Sigma_{n=1}^{\infty} \sin nx/n^3$. Justify your result.

4.(i) Let $\mathcal{F}$ be a σ-field on $\Omega = [0,1]$ such that $[\frac{1}{n+1}, \frac{1}{n}] \in \mathcal{F}$, for $n = 1, 2, \cdots$. Show that,

(a) $\{0\} \in \mathcal{F}$;

(b) $\{\frac{1}{n} \ : \ n = 2, 3, \cdots\} \in \mathcal{F}$;

(c) $(\frac{1}{n}, 1] \in \mathcal{F}$, for all n;

(d) $(0, \frac{1}{n}] \in \mathcal{F}$, for all n.

(ii) A random variable (r.v.) $X \in L_1$ satisfies $E|X| \geq a > 0$ and $EX^2 = 1$. Prove that,

$$P\{\ |X| \geq \lambda a\ \} \geq (1-\lambda)^2 a^2 \ , \ \text{for } 0 \leq \lambda \leq 1.$$

5.(i) Let $(\Omega, \mathcal{A}, P)$ be a probability space. Suppose that $\{A_n \in \mathcal{A} \ : \ n \geq 1\}$ are independent events satisfying $P(A_n) < 1$ for all $n \geq 1$. Show that

$$P(\bigcup_{n=1}^{\infty} A_n) = 1 \text{ if and only if } P(A_n \text{ i.o. }) = 1.$$

(ii) X_n are independent, with

$$P(X_n = n^2) = \frac{1}{n^2}, \quad P(X_n = -1) = 1 - \frac{1}{n^2}, \quad n = 1, 2, \cdots$$

Show that, $\lim_{n\to\infty} \sum_{k=1}^{n} X_k$ exists almost surely, on the extended real line.

6. Prove that, for an arbitrary sequence $\{X_n \ : \ n = 1, 2, \cdots\}$ of r.v.s to obey the 'weak law of large numbers' (WLLN), it is necessary and sufficient that

$$E\left(\frac{Y_n^2}{1+Y_n^2}\right) \to 0, \text{ as } n \to \infty,$$

where $Y_n := \frac{S_n}{n}$, and $S_n := \sum_{i=1}^{n} X_i$ are the partial sums.