Doctoral Qualifying Exam : Real and Complex Analysis
Monday June 12, 2008

1.

(a) Let $X = \mathbf{R}$, $\mathbf{X} = \mathbf{B}$ and the Lebesgue measure. Consider the sequence of functions $f_n(x) = \chi_{[n,n+1/n]}$. Does this sequence converge uniformly, almost everywhere, in L_p, in measure or almost uniformly? In each case prove it or explain why not.

(b) Give an example of a function that converges almost everywhere, but not in measure.

(c) Prove that almost everywhere convergence implies convergence in meausure on a finite measure space.

2. Let X by a metric space and $\mathcal{C}(X)$ be the set of all continuous and bounded functions with domain X. If $f, g \in \mathcal{C}(X)$ define the distance between these functions by

$$d(f,g) = \sup_{x \in X} |f(x) - g(x)|.$$

(a) Prove that d is a metric on C.

(b) Prove that with this definition of distance $\mathcal{C}(X)$ is a complete metric space.

3.

(a) Express the Fourier series generated by x on $[0, 2\pi]$ in the form $\Sigma c_n \phi_n$ where ϕ_n are orthonormal on the set. Clearly identify the values of c_n and ϕ_n.

(b) From (a), calculate $\Sigma_{n=1}^{\infty} 1/n^2$ using Parseval's formula.

(c) Using the above, find a function defined on $[0, 2\pi]$ whose Fourier series equals $\Sigma_{n=1}^{\infty} \sin nx/n^3$. Justify your result.

4. Compute using contour integration, with justification:

$$(a) \quad I_1 = \int_0^\infty \frac{\cos ax}{1+x^2} \quad \text{for } a > 0.$$

$$(b) \quad I_2 = \int_0^\infty \frac{\sin x}{x}\, dx.$$

5. A function $f(z)$ has a removable singularity, a pole, or an essential singularity at infinity if $f(z^{-1})$ has, respectively, a removable singularity, a pole, or an essential singularity at $z = 0$. If f has a pole at ∞ then the order of the pole is the order of the pole of $f(z^{-1})$ at $z = 0$.

(a) Prove that an entire function has a removable singularity at infinity if and only if it is a constant.

(b) Prove that an entire function has a pole at infinity of order m if and only if it is a polynomial of degree m.

6. Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : Re\ z > 0\}$. Show that this solution must be real. What happens to the solution as $\lambda \to 1$? (Hint: Consider the change in argument of $f(z) = (\lambda - z)(1 - e^{-z}/(\lambda - z)$ upon making a circuit around a large semi-circle with diameter on the imaginary axis.)