Doctoral Qualifying Exam : Linear Algebra and Numerical Methods
Monday June 9, 2008

1. Suppose that a sequence satisfies $x_{n+1} = \alpha x_n + (1-\alpha)x_{n-1}$ with $\alpha \in \mathbf{C}$.

(a) Find x_n explicitly in terms of x_0 and x_1.

(b) For what values of α does the sequence converge regardless of the initial values? What is the limiting value of the sequence when it converges?

(c) For what values of α does the sequence remain bounded regardless of the initial values?

2.

(a) Let A be an $m \times n$ complex matrix. Prove that $A^H A$ is invertible if and only if the columns of A are linearly independent.

(b) Suppose that B is a square complex matrix. Suppose that $Q = B^H B$ is unitary. Prove B is unitary.

3. Let $A \neq 0$ be an $n \times n$ complex matrix with its range (i.e. column space) contained in its kernel (i.e. null space).

(a) Prove $A^2 = 0$.

(b) Prove zero is an eigenvalue of A.

(c) Prove that zero is the only eigenvalue of A.

(d) Prove that A is defective (i.e. there is no basis for $\mathbf{C}^n$ consisting of eigenvectors of A).

(e) Prove the rank of A is less than or equal to $n/2$.

4.

(a) Write down the general iteration formula for Newton's method for rootfinding of a single equation.

(b) Suppose that you implement Newton's method in a computer code and use it to find the root of a given function. How would you determine the multiplicity of the root? (Assume that you don't know how to evaluate higher derivatives of the given function)

(c) Once you have determined the multiplicity of the root, how do you modify Newton's method to speed up the convergence?

(d) For $\alpha > 0$, let

$$f(x) = x^{1/\alpha} \quad \text{if } x \geq 0, \quad f(x) = -|x|^{1/\alpha} \quad \text{if } x < 0.$$

Apply Newton's method for any starting value. Discuss how the iteration converges or diverges, depending on the value of α.

5. Consider the following scheme for solving the differential equation $y' = f(t, y)$:

$$y_{n+2} - \frac{4}{3} y_{n+1} + \frac{1}{3} y_n = \frac{2}{3} h f(t_{n+2}, y_{n+2}).$$

(a) Find the order of this scheme.

(b) Is the scheme convergent? Is it A-stable? Explain.

6. Consider computing the integral $I(f) = \int_{-1}^{1} f(x)dx$ numerically by the n-point quadrature $Q_n(f) = w_1 f(-1) + \sum_{i=2}^{n-1} w_i f(x_i) + w_n f(1)$ where the function f is assumed to be a sufficiently smooth function.

(a) Show that $Q_n(f)$ achieves the maximal degree of precision $2n-3$ when the interior nodes $\{x_i\}$ $i = 2, \cdots, n-1$ are the zeros of $P'_{n-1}(x)$, where $P_{n-1}(x) = \frac{1}{2^{n-1}(n-1)!} \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^{n-1}$ is the Legendre polynomial of degree $n-1$.

(b) Show that all weights $\{w_i\}$ $i = 1, \cdots, n$ are positive when $Q_n(f)$ achieves the maximal degree of precision.

(c) Derive an error formula for $Q_n(f)$ which justifies its degree of precision $2n-3$.