Ph.D. Qualifying Exam: Real Analysis and Probability

Wednesday June 14, 2006

Notation. In the following problems, $\mathbf{Q}$ denotes the set of all rational numbers, and $\mathbf{R}$ denotes the real numbers. Also $L^1(S)$ and $L^2(S)$ denotes the measurable functions that are integrable and square integrable respectively over a domain S.

Problem 1. Let $\{x_1, x_2, \ldots\} = \mathbf{Q} \cap (0, 1)$. Let $H(x)$ be defined by

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Let

$$f(x) \;=\; \sum_{n=1}^{\infty} \frac{1}{2^n} H(x - x_n).$$

(a) Show that the infinite sum defining $f(x)$ converges for $x \in [0, 1]$.
(b) Show that if $\xi \in \mathbf{Q} \cap (0, 1)$ then $f(x)$ is discontinuous at ξ.
(c) Show that $f(x)$ is continuous at $x = 0$.

Problem 2. Determine which of the following functions are in $L^1(\mathbf{R})$ and which are in $L^2(\mathbf{R})$. Carefully explain all your determinations. (Assume the value of the function to be zero at any point where the function is not defined by the given formula.)

$$\text{(a)} \quad f(x) = \frac{1}{1 + |x|}$$

$$\text{(b)} \quad f(x) = \frac{e^{-|x|}}{|x|^{1/2}} \qquad\qquad \text{(c)} \quad f(x) = \sin(1/x)$$

Problem 3. For an integrable function $f : [0, 2\pi] \to \mathbf{C}$ the "sum" of its Fourier series is defined to be

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}_n e^{inx} \quad \text{where} \quad \hat{f}_n \;=\; \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx.$$

(a) Show

$$\lim_{n\to\infty} \hat{f}_n \;=\; 0.$$

(b) For each function defined below, find the sum of its Fourier series and explain why the result is valid. We denote the characteristic (or indicator) function of a set S by $\chi_S(x)$.

$$\text{(i)} \quad f(x) \;=\; \chi_A(x) \quad \text{with} \quad A = (0, 2\pi) - \mathbf{Q}$$

$$\text{(ii)} \quad f(x) \;=\; x$$

Problem 4. **(i)** Suppose $\{X_j \;:\; j = 1, 2, \cdots\}$ are *pairwise uncorrelated.* Show that, for suitable choices of centering and norming constants, either of the conditions

a) $\sum_{j=1}^{\infty}$ var $(X_j) = \infty$,

b) $n^{-2} \sum_{j=1}^{n}$ var $(X_j) \to 0$, as $n \to \infty$,

are sufficient to guarantee that the sequence $\{X_j \;:\; j = 1, 2, \cdots\}$ obeys WLLN (the Weak Law of Large Numbers).

(ii) Consider the probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}[0, 1]$, the Borel σ*-field* over the unit interval, and P is the *Lebesgue* measure on $[0, 1]$. Consider the random variables,

$$\begin{aligned} X_n : &\quad \omega \hookrightarrow \omega + \omega^n; \quad n = 1, 2, \cdots \\ X : &\quad \omega \hookrightarrow \omega \end{aligned}$$

Investigate, in what senses (among $\xrightarrow{P}$, $\xrightarrow{a.s.}$, $\xrightarrow{d}$ and $\xrightarrow{r}$), the r.v.s X_n does or, does not converge to X ?

Problem 5. Consider a Markov Chain $\{X_n :\; j = 0, 1, 2, \cdots\}$ assuming values in the state space $S = \{0, 1, 2, \cdots, N\}$ with transition probabilities $p_{ij} \equiv P(X_{n+1} = j | X_n = i)$ given by, $p_{00} = p_{NN} = 1$, and

$$p_{ij} = \begin{pmatrix} N \\ j \end{pmatrix} \left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{N-j},$$

for all $i,\, j \in S \setminus \{0, N\}$. Show that $\{X_n :\; n = 0, 1, 2, \cdots\}$ and

$$\left\{ V_n := \frac{X_n(N - X_n)}{(1 - N^{-1})^n},\; n = 0, 1, 2, \cdots \right\}$$

are both martingales. Do these martingales converge a.s. ? Why ?

6. Consider the r.v.s $X_{(n)} := \max(X_1, X_2, \cdots, X_n)$, where $X_j := F(Y_j);\; j = 1, 2, \cdots$ and $\{Y_1, Y_2, \cdots\}$ is an i.i.d. sequence with a continuous c.d.f. F.

a) Show that $X_{(n)}$ converges in probability to 1.

b) Does the convergence also hold in the r-th mean ($X_{(n)} \xrightarrow{r} 1$) for *some* or, *all* $r > 0$?

c) Find the limiting distribution of $n(1 - X_{(n)})$ as $n \to \infty$.