Ph.D Qualifying Exam: Real and Complex Analysis

Wednesday June 14, 2006

Notation. In the following problems, $\mathbf{Q}$ denotes the set of all rational numbers, and $\mathbf{R}$ denotes the real numbers. Also $L^1(S)$ and $L^2(S)$ denotes the measurable functions that are integrable and square integrable respectively over a domain S.

Problem 1. Let $\{x_1, x_2, \ldots\} = \mathbf{Q} \cap (0, 1)$. Let $H(x)$ be defined by

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} H(x - x_n).$$

(a) Show that the infinite sum defining $f(x)$ converges for $x \in [0, 1]$.
(b) Show that if $\xi \in \mathbf{Q} \cap (0, 1)$ then $f(x)$ is discontinuous at ξ.
(c) Show that $f(x)$ is continuous at $x = 0$.

Problem 2. Determine which of the following functions are in $L^1(\mathbf{R})$ and which are in $L^2(\mathbf{R})$. Carefully explain all your determinations. (Assume the value of the function to be zero at any point where the function is not defined by the given formula.)

(a) $\quad f(x) = \dfrac{1}{1 + |x|}$

(b) $\quad f(x) = \dfrac{e^{-|x|}}{|x|^{1/2}}$ (c) $\quad f(x) = \sin(1/x)$

Problem 3. For an integrable function $f : [0, 2\pi] \to \mathbf{C}$ the "sum" of its Fourier series is defined to be

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}_n e^{inx} \qquad \text{where} \qquad \hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx.$$

(a) Show

$$\lim_{n\to\infty} \hat{f}_n = 0.$$

(b) For each function defined below, find the sum of its Fourier series and explain why the result is valid. We denote the characteristic (or indicator) function of a set S by $\chi_S(x)$.

$$\text{(i)} \quad f(x) \;=\; \chi_A(x) \qquad \text{with } A = (0, 2\pi) - \mathbf{Q}$$

$$\text{(ii)} \quad f(x) \;=\; x$$

Problem 4. Evaluate the following integrals:

$$(a) \int_0^\infty \frac{\sin x}{x(1+x^2)}\, dx, \quad (b) \int_0^\infty \frac{\ln x}{x^4+1}\, dx$$

In (b), $\ln 1 = 0$.

Problem 5. Show that the function

$$f(z) = 2z^5 + 7z - 1$$

has a real positive root in $|z| < 1$ and that the remaining four roots lie in $1 < |z| < 2$.

Problem 6. Suppose that $f(z)$ is analytic in a domain D, and that $f(z)$ is bounded in D, i.e., $|f(z)| \leq M$ for all z in D, where M is a positive number. Show that for all z in D

$$|f'(z)| \leq \frac{M}{d(z)}.$$

where $d(z)$ is the distance from the point z to the boundary of D.